Chapter 2
Order-Preserving Maps

The notion of “monotonicity” of a function arises in various garbs in almost all branches of mathematics. Roughly speaking, and in the abstract, a monotonic function is one that maps one ordered structure to another in such a way that the ordering of any two elements in the domain is either inherited or reversed by the images of those elements in the codomain. In this chapter, we examine this notion at length, and consider a number of additional constructs of order theory that the property of “monotonicity” naturally leads us. In particular, we provide here introductory accounts of Galois connections, the Dedekind-MacNeille completion of a poset, and order-preserving set-valued maps.

1 Order-Preserving Functions

1.1 Definitions and Examples

Our immediate objective is to extend the familiar notion of “monotonicity” of a self-map on $\mathbb{R}$ to the context of maps between two preordered sets.

**Definition.** Let $(X, \preceq_X)$ and $(Y, \preceq_Y)$ be two preordered sets. By an **order-preserving map** from $(X, \preceq_X)$ into $(Y, \preceq_Y)$, we mean a function $f : X \to Y$ such that

$$x \preceq_X y \quad \text{implies} \quad f(x) \preceq_Y f(y),$$

for every $x$ and $y$ in $X$. Dually, a function $f : X \to Y$ is called **order-reversing** from $(X, \preceq_X)$ into $(Y, \preceq_Y)$ if it is an order-preserving map from $(X, \preceq_X)$ into $(Y, \preceq_Y)$.

**Note.** The terminology is not entirely uniform in the literature. In particular, some authors use the term “isotonic” for “order-preserving.”
Note. A function \( f : X \to Y \) is an order-preserving map from a preordered set \((X, \preceq_X)\) into another \((Y, \preceq_Y)\) iff \( f^{-1}(S) \) is a \( \preceq_X \)-decreasing set in \( X \) for every \( \preceq_Y \)-decreasing subset \( S \) of \( Y \).

As usual, we sometimes adopt a few terminological shortcuts in our exposition. In particular, when the preorders \( \preceq_X \) and \( \preceq_Y \) are understood from the context, we refer to an “order-preserving map from \((X, \preceq_X)\) into \((Y, \preceq_Y)\),” simply as “order-preserving,” and similarly for “order-reversing” maps. Furthermore, if \( X = Y \), and \( \preceq_X \) and \( \preceq_Y \) are the same preorder, say, \( \preceq \), we refer to an “order-preserving (order-reversing) map from \((X, \preceq)\) into \((X, \preceq)\),” simply as an \( \preceq \)-preserving (\( \preceq \)-reversing) self-map on \( X \).

**Definition.** Let \((X, \preceq_X)\) and \((Y, \preceq_Y)\) be two preordered sets. By an order-embedding from \((X, \preceq_X)\) into \((Y, \preceq_Y)\), we mean a function \( f : X \to Y \) such that

\[
x \preceq_X y \text{ if and only if } f(x) \preceq_Y f(y)
\]

for every \( x \) and \( y \) in \( X \). If there exists such a function, we say that \((X, \preceq_X)\) can be order-embedded in \((Y, \preceq_Y)\).

**Note.** An order-embedding \( f \) from a preordered set \((X, \preceq_X)\) into another preordered set \((Y, \preceq_Y)\) is injective up to equivalence \( \sim_X \), that is, \( f(x) = f(y) \) implies \( x \sim_X y \) for every \( x, y \in X \). In particular, any order-embedding from a poset into a preordered set is an injection.

If a preordered set can be order-embedded in another preordered set, we can think of the latter containing a copy of the former one. So, insofar as order theory is concerned, the latter preordered set has more content than the former one. Consequently, if we can also order-embed the latter in the former, we must conclude that these two preordered sets have the same order-theoretic content. This leads us to the following:

**Definition.** Let \((X, \preceq_X)\) and \((Y, \preceq_Y)\) be two preordered sets. A bijective order-embedding from \((X, \preceq_X)\) into \((Y, \preceq_Y)\) is said to be an order-isomorphism from \((X, \preceq_X)\) onto \((Y, \preceq_Y)\). If there exists such a function, we say that \((X, \preceq_X)\) and \((Y, \preceq_Y)\) are order-isomorphic.

To reiterate, the order-theoretic properties of order-isomorphic preordered sets are identical. Put differently, in the context of order theory, we can regard two order-isomorphic preordered sets as the “same,” for either one of these preordered sets can be obtained from the other by merely relabeling its elements.

The following examples provide some illustrations of these definitions.
Example 1.1.1. Let \((X, \succeq)\) be a preordered set. The identity function on \(X\), and any constant function on \(X\) (that is, a function that maps every element of \(X\) to a fixed element of \(X\)) is a \(\succeq\)-preserving self-map on \(X\). Such functions are called trivial \(\succeq\)-preserving self-maps on \(X\). Among these maps, only the identity function is an order-embedding, unless we have \(x \sim y\) for every \(x\) and \(y\) in \(X\).

Example 1.1.2. Let \((X, \succ)\) be a finite loset, and set \(n := |X|\). Then, \((X, \succ)\) and \(([n], \succeq)\), are order-isomorphic. (Recall that \([n] := \{1, \ldots, n\}\) and \(\succeq\) is the usual ordering of integers.) To see this formally, we let \(x_1\) be the \(\succ\)-minimum of \(X\) and \(x_i\) the \(\succ\)-minimum of \(X\setminus\{x_1, \ldots, x_{i-1}\}\) for each \(i = 2, \ldots, n\) – recall Proposition 5.1 of Chapter 1 – and define \(f : X \to [n]\) by \(f(x_i) := i\). It is plain that \(f\) is an order-isomorphism from \((X, \succ)\) onto \(([n], \succeq)\).

Example 1.1.3. A function \(f : \mathbb{R}^n \to \mathbb{R}^n\) is a \(\succeq\)-preserving self-map if it is increasing in each of its components. For instance, where \(A\) is a nonnegative \(n \times n\) matrix, the map \(x \mapsto Ax\) is \(\succeq\)-preserving. If \(A\) is invertible and \(A^{-1}\) nonnegative, this map is an order-isomorphism.

Example 1.1.4. Consider the self-map \(f\) on \(\mathbb{N}\) defined by \(f(i) := i + 1\). Then, \(f\) is an order-embedding from \((\mathbb{N}, \succeq)\) into \((\mathbb{N}, \succeq)\), but it is not an order-isomorphism.

Example 1.1.5. The intervals \([0, 1]\) and \([0, 1)\) are not order-isomorphic. For, \([0, 1]\) has a \(\succeq\)-maximum but \([0, 1)\) does not. Similarly, \(\mathbb{Q}\) and \(\mathbb{N}\) are not order-isomorphic. (How about \(\mathbb{Q}\) and \(\mathbb{Z}\)?)

Example 1.1.6. Let \(X\) be a topological space, and consider the self-map \(f\) on \(2^X\) defined by \(f(S) := \text{cl}(S)\). Then, \(f\) is a \(\succeq\)-preserving self-map. This function is not an order-embedding in general. (After all, \(f\) is not even injective. For instance, the closure of \(\mathbb{Q}\) and \(\mathbb{R}\) both equal \(\mathbb{R}\).)

Example 1.1.7. Let \(X\) and \(Y\) be two (real) linear spaces, and \(C\) and \(D\) convex cones in \(X\) and \(Y\), respectively. (Recall Example 3.1.7 of Chapter 1.) Let \(f : X \to Y\) be a linear map, that is, \(f(\lambda x) = \lambda f(x)\) and \(f(x + y) = f(x) + f(y)\) for every \(x\) and \(y\) in \(X\) and real number \(\lambda\). Then, \(f\) is an order-preserving map from \((X, \succeq_C)\) into \((Y, \succeq_D)\) iff \(f(C) \subseteq D\). Indeed, if \(f\) is order-preserving and \(x \in C\), we have \(x \succeq_C 0_X\), and hence

\[ f(x) \succeq_D f(0_X) = 0_Y, \]
that is, \( f(x) \in D \). (Here \( 0_X \) is the origin of \( X \), and \( 0_Y \) is that of \( Y \).) Conversely, if \( f(C) \subseteq D \) and \( x \geq_C y \), then \( x - y \in C \), and hence

\[
f(x) - f(y) = f(x - y) \in f(C) \subseteq D,
\]
that is, \( f(x) \geq_D f(y) \).

Similarly, if \( f \) is injective and \( f(C) = D \), it is easily seen that \( f \) is an order-embedding from \((X, \geq_C)\) into \((Y, \geq_D)\). Moreover, provided that it is bijective, \( f \) is an order-isomorphism iff \( f(C) = D \).

**Example 1.1.8.** Let \( X \) be a nonempty set, \( \mathcal{E}_X \) the collection of all equivalence relations on \( X \), and \( \mathcal{P}_X \) the set of all partitions of \( X \). We have seen in Example 8.1.7 of Chapter 1 that \((\mathcal{E}_X, \supseteq)\) is a complete lattice. Proposition 2.1 of that chapter suggests that we can also order \( \mathcal{P}_X \) to obtain a poset that would be “equivalent” to \((\mathcal{E}_X, \supseteq)\). To this end, we define the binary relation \( \succ \) on \( \mathcal{P}_X \) as \( A \succ B \) iff for every \( A \) in \( \mathcal{A} \) there is a \( B \) in \( \mathcal{B} \) with \( A \supseteq B \). We wish to show that the posets \((\mathcal{E}_X, \supseteq)\) and \((\mathcal{P}_X, \succ)\) are order-isomorphic. To this end, consider the map \( f : \mathcal{E}_X \to \mathcal{P}_X \), defined by

\[
f(\sim) := X/\sim.
\]

Notice that an equivalence relation \( \sim \) on \( X \) is a superset of another such relation \( \approx \) iff \( [x]_\sim \supseteq [x]_\approx \) for every \( x \in X \), that is,

\[
\sim \supseteq \approx \iff X/\sim \supseteq X/\approx,
\]
which means that \( f \) is an order-embedding from \((\mathcal{E}_X, \supseteq)\) into \((\mathcal{P}_X, \succ)\). On the other hand, for any partition \( \mathcal{A} \) of \( X \), the binary relation on \( X \), defined by \( x \sim y \) iff \( \{x, y\} \subseteq A \) for some \( A \in \mathcal{A} \), is an equivalence relation on \( X \) with \( X/\sim = \mathcal{A} \), that is, \( f \) is a surjection.

Conclusion: \( f \) is an order-isomorphism from \((\mathcal{E}_X, \supseteq)\) onto \((\mathcal{P}_X, \succ)\), and hence, \((\mathcal{E}_X, \supseteq)\) and \((\mathcal{P}_X, \succ)\) are order-isomorphic. As an immediate consequence of this fact, we find that \((\mathcal{P}_X, \succ)\) is a complete lattice.

### 1.2 \( \lor \)- and \( \land \)-Preservation

An order-preserving function maps the extrema to extrema. Put more precisely:

**Proposition 1.2.1.** Let \( f \) be an order-preserving map from a preordered set \((X, \preceq_X)\) into another preordered set \((Y, \preceq_Y)\). Then,

\[
f(\max(S, \preceq_X)) \subseteq \max(f(S), \preceq_Y)
\]
for every nonempty subset \(S\) of \(X\). On the other hand, if \(f\) is an order-isomorphism, we have

\[
f(\text{MAX}(S, \preceq_X)) = \text{MAX}(f(S), \preceq_Y)
\]

for every nonempty subset \(S\) of \(X\).

This is a trivial observation. However, it is important to note that the containment above may hold strictly. (Consider, for instance, the constant map \(x \mapsto 1\) on \([0, 1]\).)

**Proposition 1.2.2.** Let \(f\) be an order-preserving map from a poset \((X, \preceq_X)\) into another poset \((Y, \preceq_Y)\). Then, for any subset \(S\) of \(X\),

\[
f(\bigvee S) \preceq_Y \bigvee f(S)
\]

provided that \(\bigvee S\) and \(\bigvee f(S)\) exist (in \(X\) and \(Y\), respectively). Similarly,

\[
\bigwedge f(S) \succeq_Y f(\bigwedge S)
\]

provided that \(\bigwedge S\) and \(\bigwedge f(S)\) exist (in \(X\) and \(Y\), respectively).

**Proof.** Let \(S\) be a subset of \(X\) such that \(\bigvee S\) and \(\bigvee f(S)\) exist. Obviously, \(\bigvee S \succeq_X S\), so, as \(f\) is order-preserving, we have \(f(\bigvee S) \succeq_Y f(S)\). That is, \(f(\bigvee S)\) is an \(\succeq_Y\)-upper bound for \(f(S)\). But, by definition, \(\bigvee f(S)\) is the \(\succeq_Y\)-minimum of such upper bounds, so (1) must be true. The second assertion follows from duality.

Once again, the inequalities in this proposition may well hold strictly.

**Example 1.2.1.** Let \(X\) be the subset of \(\mathbb{R}^2\) that consists of \((1, 0)\), \((0, 1)\), \((1, 1)\) and \((2, 2)\). Consider the self-map \(f\) on \(X\) that maps \((1, 0)\) and \((0, 1)\) to \((1, 1)\), and \((1, 1)\) and \((2, 2)\) to \((2, 2)\). Then, \(f\) is \(\succeq\)-preserving, but for \(S := \{(1, 0), (0, 1)\}\), we have

\[
f(\bigvee S) = (2, 2) > (1, 1) = \bigvee f(S).
\]

**Example 1.2.2.** Define the self-map \(f\) on \(2^\mathbb{R}\) by \(f(S) := \text{cl}(S)\). Then, \(f\) is \(\supseteq\)-preserving, but where \(Q := \{\{r\} : r \in \mathbb{Q}\}\), we have

\[
f(\bigvee Q) = f(\bigcup Q) = f(Q) = \text{cl}(Q) = \mathbb{R}
\]

while

\[
\bigvee f(Q) = \bigcup \{f(\{r\}) : r \in \mathbb{Q}\} = \mathbb{Q}.
\]
Of course, all of the inequalities in Proposition 1.2.2 hold as equalities when the involved function is an order-isomorphism. That is:

**Proposition 1.2.3.** Let \( f \) be an order-isomorphism from a poset \((X, \triangleright_X)\) onto another poset \((Y, \triangleright_Y)\). Then, \( f \) is both \( \lor \)- and \( \land \)-preserving, that is,

\[
f(\lor S) = \lor f(S) \quad \text{and} \quad f(\land S) = \land f(S),
\]

for every subset \( S \) of \( X \) such that the involved suprema and infima exist.

The proof of this result is easy, and hence omitted. But we should note that the \( \lor \)- and \( \land \)-preservation properties considered in Proposition 1.2.3 are essential for lattice theory, and prompt the following:

**Definition.** Let \((X, \triangleright_X)\) and \((Y, \triangleright_Y)\) be two lattices. A function \( f : X \to Y \) is said to be a **lattice homomorphism** from \((X, \triangleright_X)\) into \((Y, \triangleright_Y)\) if

\[
f(x \lor y) = f(x) \lor f(y) \quad \text{and} \quad f(x \land y) = f(x) \land f(y),
\]

for every \( x \) and \( y \) in \( X \).

**Note.** By induction, we see that the supremum and infimum of any finite set in a lattice are preserved under a lattice homomorphism. This is, however, not true for infinite sets in general. For instance, where \( X := [0,1) \cup \{2\} \) and \( Y := [0,1] \cup \{2\} \), the map \( f : X \to Y \), defined by \( f(x) := x \), is a lattice homomorphism from \((X, \geq)\) into \((Y, \geq)\), but \( f(\lor [0,1)) = f(2) = 2 \) while \( \lor f((0,1)) = \lor [0,1) = 1 \).

It is plain that every lattice homomorphism is order-preserving, but not conversely. It is also easily verified that an injective lattice homomorphism is an order-embedding, but not conversely. In summary:

\[
\text{lattice homomorphism} \implies \text{order-preserving map}
\]

and

\[
\text{injective lattice homomorphism} \implies \text{order-embedding}
\]

Yet, in the case of bijections, the \( \lor \)- and \( \land \)-preservation properties and that of being an order-embedding are one and the same:

\[
\text{bijective lattice homomorphism} \iff \text{order-isomorphism}
\]

As we shall use this fact below frequently, we state it more explicitly:
**Proposition 1.2.4.** A function $f : X \to Y$ is a bijective lattice homomorphism from a lattice $(X, \succsim_X)$ into another lattice $(Y, \succsim_Y)$ if, and only if, it is an order-isomorphism from $(X, \succsim_X)$ into $(Y, \succsim_Y)$.

Once again we leave the (easy) proof as an exercise.

### 1.3 Order-Embeddability

There is a natural way of order-embedding any given poset $(X, \succsim)$ in a lattice of sets, for instance, in the lattice of all $\succsim$-decreasing subsets of $X$. Indeed, if we identify each member $x$ of $X$ with $x^\dagger$, the resulting collection of sets, under the usual containment ordering, would be a poset that is order-isomorphic to our original poset. More generally, we have:

**Proposition 1.3.1.** Let $(X, \succeq)$ be a preordered set. Then, the map $f : X \to 2^X$, defined by

$$f(x) := x^\dagger,$$

is an order-embedding from $(X, \succeq)$ into $2^X$.

**Proof.** Transitivity of $\succeq$ ensures that $x \succeq y$ implies $f(x) \supseteq f(y)$, while its reflexivity guarantees the converse of this implication hold true.

**Note.** When $(X, \succeq)$ is a poset, the map $f$ considered in this result is called the canonical order-embedding. This map has some desirable properties. In particular, it is $\bigwedge$-preserving (provided we adopt the convention that $\bigcap \emptyset = X$). It is, however, not $\bigvee$-preserving in general.

**Corollary 1.3.2.** Every preordered set $(X, \succeq)$ can be order-embedded in a complete lattice of sets, namely, $2^X$.

Given Proposition 1.3.1, it is natural to ask if a poset can ever be order-isomorphic to the lattice of its order-decreasing subsets. It turns out that this is impossible. In fact, a one-to-one correspondence between a poset and the lattice of its order-decreasing subsets can never be order-preserving. This (and more) is the content of the following important result:

**The Gleason-Dilworth Theorem.** Let $(X, \succsim)$ be a poset and $f : 2^X \to X$ an injection. Then, neither $f$ nor $f^{-1}$ (defined on $f(2^X)$) can be order-preserving.
Proof. To derive a contradiction, let us assume that $f$ is order-preserving. We define

$$
\mathcal{A} := \{ A \in 2^X : f(A) \subseteq A \},
$$

and let $A_* := \bigcap \mathcal{A}$. Then, for any $A$ in $\mathcal{A}$, we have $f(A) \supseteq f(A_*)$, because $f$ is order-preserving. It follows that $f(A_*) \subseteq A$, because $f(A) \subseteq A$ and $A$ is $\supseteq$-decreasing. Conclusion: $f(A_*) = f(A_*)$, that is, $A_* \subseteq A$. Now, for any $x$ in $\downarrow f(A_*)$, we have

$$
A \ni f(A) \supseteq f(A_*) \supseteq x \quad \text{for every } A \in \mathcal{A},
$$

and hence $x \in A_*$. Thus: $\downarrow f(A_*) \subseteq A_*$. As $\downarrow f(A_*)$ does not contain $f(A_*)$, but $A_*$ does, therefore, $\downarrow f(A_*) \subset A_*$. It follows that $f(A_*) \supseteq f(\downarrow f(A_*))$, because $f$ is injective and order-preserving. But then

$$
f(\downarrow f(A_*)) \subseteq \downarrow f(A_*),
$$

which means $\downarrow f(A_*) \in \mathcal{A}$, and hence $A_* \subseteq \downarrow f(A_*)$, contradiction.

To prove the second part of the assertion, assume $f^{-1}$ is order-preserving, and define

$$
\mathcal{B} := \{ B \in 2^X : f(B) \subseteq X \setminus B \}
$$

and

$$
\mathcal{S} := \{ x \in X : f(B) \supseteq x \text{ for some } B \in \mathcal{B} \}.
$$

Then, if $f(S) \in \mathcal{S}$, we find $f(B) \supseteq f(S)$, and hence $B \supseteq S$ (because $f^{-1}$ is order-preserving) for some $B$ in $\mathcal{B}$. As we have $f(B) \in \mathcal{S}$ by definition of $\mathcal{S}$, this implies $f(B) \in B$, which contradicts $B$ being a member of $\mathcal{B}$. Assume, then, $f(S) \in X \setminus \mathcal{S}$. Since $\mathcal{S} \subseteq 2^X$, then, we have $S \in \mathcal{B}$, and hence, by reflexivity of $\supseteq$, $f(S) \in S$, a contradiction.

Note. The first part of the Gleason-Dilworth Theorem is not true for preordered sets in general. For instance, relative to the preordered set $(X, X \times X)$, where $X := \{ x, y \}$, mapping $x$ to $\emptyset$ and $y$ to $X$ yields an order-preserving bijection from $(X, X \times X)$ onto $2^X$. The second part of the result, however, remains true for preordered sets as well.

Remark. The second part of the Gleason-Dilworth Theorem generalizes a famous theorem of Georg Cantor: There is no surjection from a set $X$ onto $2^X$. (This entails that the power set of any set has strictly higher cardinality than that set, thereby proving that the set of all sets is not a “set,” the so-called Cantor’s Paradox.) Indeed, if we take $\supseteq$ as $\Delta_X$ in the Gleason-Dilworth Theorem, then $2^X = 2^X$ and every $f : 2^X \to X$ is order-preserving. Thus, in this special case, the result above tells us simply that there does not exist an injection from $2^X$ into $X$, which means that there is no surjection from a subset of $X$ onto $2^X$. 

8
1.1. Prove that \((\mathbb{Z}, \ge)\) and \((\mathbb{Q}, \ge)\) are not order-isomorphic. How about \((\mathbb{R}, \ge)\) and \((\mathbb{Q}, \ge)\)?

1.2. Let \(X\) be a nonempty finite set. Prove that \((2^X, \supseteq)\) and \(((0, 1)^{|X|}, \ge)\) are order-isomorphic.

1.3. Prove: For any poset \((X, \gg)\), the posets \((2^X, \supseteq)\) and \(((0, 1)^{|X|}, \ge)\) are order-isomorphic.

1.4. Let \(X\) be a nonempty set, and take any two nonempty subsets \(S\) and \(T\) of \(X\) with \(S \subseteq T\). Let \(\mathcal{X} := \{A \in 2^X : S \subseteq A \subseteq T\}\). Prove that \((\mathcal{X}, \supseteq)\) and \((2^{T \setminus S}, \supseteq)\) are order-isomorphic.

1.5. Let \(f : X \rightarrow Y\) be an order-isomorphism from a lattice \((X, \gg_X)\) into another lattice \((Y, \gg_Y)\). Show that the map \(F : 2^X \rightarrow 2^Y\), defined by \(F(S) := \bigcup\{f(x) : x \in S\}\), is also an order-isomorphism.

1.6. Let \(n\) be a positive integer, and consider the poset \((\text{Sym}_n, \gg)\) introduced in Exercise 3.3 of Chapter 1. Let \(\text{Sym}_n^+\) denote the collection of all positive semidefinite \(n \times n\) matrices, and define \(f : \text{Sym}_n^+ \rightarrow \text{Sym}_n\) by

\[
f(Q) := X + \sum_{i=1}^{m} A_i^T \phi(Q) A_i,
\]

where \(X \in \text{Sym}_n^+\), \(\phi\) is a \(\gg\)-preserving self-map on \(\text{Sym}_n^+\), and \(A_1, ..., A_m\) are arbitrary \(n \times n\) matrices. (Here \(A_i^T\) denotes the transpose of \(A_i\) for each \(i\).) Show that \(f\) is a \(\gg\)-preserving self-map on \(\text{Sym}_n^+\) with \(f(X) \gg X\).

1.7. Let \(n\) be a positive integer, and recall that every symmetric \(n \times n\) matrix has at least one real eigenvalue. Consider the map \(f : \text{Sym}_n \rightarrow \mathbb{R}\) defined by

\[
f(A) := \lambda_{\text{min}}(A) + \lambda_{\text{max}}(A),
\]

where \(\lambda_{\text{min}}(A)\) and \(\lambda_{\text{max}}(A)\) are the minimum and maximum real eigenvalues of \(A\). Prove that \(f\) is an order-isomorphism from \((\text{Sym}_n, \gg)\) onto \(\mathbb{R}\).

1.8. For any positive integer \(n\), let \(\mathcal{X}_n\) stand for the collection of all linear subspaces of \(\mathbb{R}^n\). Prove that \((\mathcal{X}_n, \supseteq)\) and \((\mathcal{X}_n, \subseteq)\) are order-isomorphic.

1.9. Prove: Two finite posets are order-isomorphic iff one’s Hasse diagram is also the Hasse diagram of the other.

1.10. Let \(f\) be an order-isomorphism from a poset \((X, \gg_X)\) onto another poset \((Y, \gg_Y)\). Define the equivalence relation \(\sim_f\) on \(X\) by \(x \sim_f y\) iff \(f(x) = f(y)\), and consider the map \(F : X/\sim_f \rightarrow Y\) defined by \(F([x]_{\sim_f}) := f(x)\). Prove that \(F\) is a (well-defined) order-isomorphism from the poset \((X/\sim_f, \supseteq)\) onto \((Y, \gg_Y)\), where \(\supseteq\) is defined as \([x]_{\sim_f} \supseteq [y]_{\sim_f}\) iff \(x \gg_X y\).

1.11. Prove that every order-isomorphism from \(\mathbb{R}\) onto \(\mathbb{R}\) is continuous.

1.12. Find two posets such that the first one can be order-embedded in the second one, and vice versa, while these posets are not order-isomorphic.

1.13. As we have seen in Exercise 8.4 of Chapter 1, \((\text{Co}(X, \gg), \supseteq)\) is a complete lattice for any poset \((X, \gg)\).

   a. Prove that \((\text{Co}(X, \gg), \supseteq)\) and \((\text{Co}(X, \ll), \supseteq)\) are order-isomorphic.

   b. Let \(x\) be an element of \(X\) with \(x \ll y\) for each \(y\) in \(X \setminus \{x\}\). Prove that \((\text{Co}(X, \gg), \supseteq)\) is order-isomorphic to the product of \((\text{Co}(X \setminus \{x\}, \gg), \supseteq)\) and \(([2], \ge)\).
1.14. Let \( f : X \to Y \) be an order-preserving map from a poset \((X, \triangleright_X)\) into another poset \((Y, \triangleright_Y)\). Prove: \( f(S) \uparrow = f(S \uparrow) \) for any subset \( S \) of \( X \).

1.15. Let \((X, \triangleright)\) be a poset. Prove that the map \( f : X \to 2^X \) defined by \( f(x) := x \uparrow \), is \( \wedge \)-preserving (provided that we adopt the convention that \( \bigcap \emptyset = X \)), but it need not be \( \vee \)-preserving.

1.16. Let \( f : X \to Y \) be a lattice-homomorphism from a lattice \((X, \triangleright_X)\) into another lattice \((Y, \triangleright_Y)\). Does \((f(X), \triangleright_Y)\) have to be a sublattice of \((Y, \triangleright_Y)\)? What if \( f \) is injective?

1.17. Let \( \triangleright \) and \( \trianglerighteq \) be two partial orders on a nonempty set \( X \) such that a self-map on \( X \) is both \( \triangleright \)-preserving and \( \trianglerighteq \)-preserving iff it is trivial. Show that \( \triangleright \cap \trianglerighteq = \triangle_X \).

1.18. Let \((X, \triangleright)\) be a poset and \( Y \) a subset of \( X \) such that \((Y, \triangleright)\) is a complete lattice. Show that there is an order-preserving map \( f \) from \((X, \triangleright)\) into \((Y, \triangleright)\) such that \( f(y) = y \) for every \( y \in Y \).

1.19. Let \((X, \triangleright)\) be a complete lattice and \( f \) a \( \triangleright \)-preserving self-map on \( X \). Let \( Y := \{ x \in X : x \triangleright f(x) \} \), and show that \((Y, \triangleright)\) is a complete lattice. Also, give an example to show that \((Y, \triangleright)\) need not be a sublattice of \((X, \triangleright)\).

1.20. Prove or disprove: For any poset \((X, \triangleright)\), there is no surjection from a set \( X \) onto \( 2^X \).

2 Galois Connections

We have noted above that two posets that are order-isomorphic are identical from the order-theoretic viewpoint. This is the strongest way in which two posets can be related to each other. Of course, there are weaker ways in which we can do this as well. For instance, two posets can be related to each other by means of a \( \vee \)-preserving bijection, which means that insofar as the suprema of their subsets are concerned, these two posets are identical. In this section, we shall outline a more subtle, but even more useful, manner in which two posets can be related to each other.

2.1 Definitions and Examples

The notion of Galois connections was introduced to order theory by Øystein Ore in 1944. The following is a somewhat more modern formulation of this concept (due to Schmidt (1956)).

**Definition.** Let \((X, \preceq_X)\) and \((Y, \preceq_Y)\) be two preordered sets, and \( f : X \to Y \) and \( F : Y \to X \) two functions. The ordered pair \((f, F)\) is said to be a Galois connection between \((X, \preceq_X)\) and \((Y, \preceq_Y)\) if

\[
y \preceq_Y f(x) \iff F(y) \preceq_X x
\]

for every \( x \) in \( X \) and \( y \) in \( Y \). In this case, we say that \( f \) is a lower adjoint, and \( F \) is an upper adjoint, between \((X, \preceq_X)\) and \((Y, \preceq_Y)\).
Note. \( (f, F) \) is a Galois connection between \((X, \succeq_X)\) and \((Y, \succeq_Y)\) iff \((F, f)\) is a Galois connection between \((Y, \preceq_Y)\) and \((X, \preceq_X)\). This is the **duality principle for Galois connections**.

Note. The terms “lower adjoint” and “upper adjoint” are hardly self-explanatory. Suffice it to say here that the roots of these terms come from category theory. (See Erné et al. (2006) for more on this.)

Here is our first example.

**Example 2.1.1.** Let \( f \) be an order-isomorphism from a preordered set \((X, \succeq_X)\) onto another preordered set \((Y, \succeq_Y)\). Then, \((f, f^{-1})\) is a Galois connection between \((X, \succeq_X)\) and \((Y, \succeq_Y)\).

Therefore, being a lower adjoint is a less stringent requirement than being an order-isomorphism:

\[
\boxed{\text{order-isomorphism}} \implies \boxed{\text{lower adjoint}}
\]

It is easy to see that the converse implication is false, that is, two preordered sets that are not order-isomorphic may well admit a Galois connection. In fact, it is even true that a loset can be Galois connected to a poset with an arbitrarily large width.

**Example 2.1.2.** Define the maps \( f : \mathbb{R} \to \mathbb{R}^n \) and \( F : \mathbb{R}^n \to \mathbb{R} \) by

\[
f(x) := (x, \ldots, x) \quad \text{and} \quad F(x) := \min\{x_1, \ldots, x_n\}.
\]

It is an easy exercise to check that \((f, F)\) is a Galois connection between \(\mathbb{R}\) and \(\mathbb{R}^n\).

**Note.** A straightforward reformulation of the previous example allows us to look at the binary supremum and infimum operations on a lattice as lower and upper adjoints between that lattice and the product of two copies of that lattice, respectively. Indeed, if \((X, \triangleright)\) is a lattice, and the functions \( f : X \to X \times X \) and \( F : X \times X \to X \) are defined by \( f(x) := (x, x) \) and \( F(x, y) := \wedge\{x, y\} \), then \((f, F)\) is a Galois connection between \((X, \triangleright)\) and the product of \((X, \triangleright)\) with itself.

As a matter of fact, instances of Galois connections abound across various branches of mathematics. As we shall see shortly, this allows one to use them to unify a large number of ostensibly different looking phenomena. We illustrate this next by means of several examples.

We begin with two examples from elementary set theory.

**Example 2.1.3.** Let \( X \) and \( Y \) be two nonempty sets, and \( f : X \to Y \) and \( F : Y \to X \) two functions. Then, \((f, F)\) is a Galois connection between \((X, \triangle_X)\) and \((Y, \triangle_Y)\) iff
$F \circ f = \text{id}_X$ and $f \circ F = \text{id}_Y$. Put differently, $f$ is a lower adjoint between $(X, \triangle_X)$ and $(Y, \triangle_Y)$ iff $f$ is invertible and $f^{-1} = F$.

**Example 2.1.4.** Let $X$ be a nonempty set, and $\varphi$ a self-map on $X$. Define the self-maps $f$ and $F$ on $2^X$ by

$$f(S) := \varphi(S) \quad \text{and} \quad F(T) := \varphi^{-1}(T).$$

(Here $\varphi^{-1}(T)$ stands for the inverse image of $T$ under $\varphi$, that is, it equals $\{x \in X : \varphi(x) \in T\}$.) It is easily checked that $(f, F)$ is a Galois connection between $(2^X, \supseteq)$ and itself.

Our next example is from topology.

**Example 2.1.5.** Let $X$ be a topological space, and recall that we denote the collection of all open subsets of $X$ by $\mathcal{O}_X$ and that of all closed subsets of $X$ by $\mathcal{C}_X$. Define the maps $f : \mathcal{O}_X \to \mathcal{C}_X$ and $F : \mathcal{C}_X \to \mathcal{O}_X$ by

$$f(O) := \text{cl}(O) \quad \text{and} \quad F(C) := \text{int}(C).$$

Then, $(f, F)$ is a Galois connection between $(\mathcal{O}_X, \supseteq)$ and $(\mathcal{C}_X, \supseteq)$. Indeed, for any $C$ in $\mathcal{C}_X$ and $O$ in $\mathcal{O}_X$, it is obvious that $C \supseteq \text{cl}(O)$ implies that $C$ contains $O$. But then, since $O$ is open, we have $\text{int}(C) \supseteq O$. Conversely, if $\text{int}(C) \supseteq O$, then

$$C = \text{cl}(C) \supseteq \text{cl}(\text{int}(C)) \supseteq \text{cl}(O).$$

It follows that $C \supseteq \text{cl}(O)$ iff $\text{int}(C) \supseteq O$, as we sought.

The following examples are from linear algebra and convex analysis, respectively.

**Example 2.1.6.** Let $(X, \langle \cdot, \cdot \rangle)$ be an inner product space, and recall that the **orthogonal complement** of a set $S$ in this space is defined as

$$S^\perp := \{ \omega \in X : \langle x, \omega \rangle = 0 \text{ for each } x \in S \}.$$ 

It is easy to show that $S^\perp$ is a closed linear subspace of $X$ and we have $S \subseteq S^\perp$, for every $S \subseteq X$.\footnote{As usual, by a closed linear subspace of $X$, we mean a linear subspace of $X$ which is closed relative to the metric $(x, y) \mapsto \langle x - y, x - y \rangle^{1/2}$ on $X$.} It is also plain that $S \subseteq T$ implies $S^\perp \supseteq T^\perp$ for any two subsets of $S$ and $T$ of $X$.\footnote{As usual, by a closed linear subspace of $X$, we mean a linear subspace of $X$ which is closed relative to the metric $(x, y) \mapsto \langle x - y, x - y \rangle^{1/2}$ on $X$.}
Now, let $\mathcal{X}$ be the collection of all closed linear subspaces of $X$, and define the maps $f : 2^X \to \mathcal{X}$ and $F : \mathcal{X} \to 2^X$ as

$$f(S) := S^\perp \quad \text{and} \quad F(Y) := Y^\perp.$$ 

Then, $(f, F)$ is a Galois connection between $(2^X, \supseteq)$ and $(\mathcal{X}, \subseteq)$. Indeed, if $Y \subseteq f(S)$ for some subset $S$ of $X$ and a closed linear subspace $Y$ of $X$, then $Y^\perp \supseteq f(S)^\perp$, and hence

$$F(Y) = Y^\perp \supseteq S^{\perp\perp} \supseteq S.$$

That $F(Y) \supseteq S$ implies $Y \subseteq f(S)$ is similarly proved.

**Example 2.1.7.** For any nonempty subset $S$ of $\mathbb{R}^n$, we define the **polar** of $S$ as

$$S^\circ := \{ c \in \mathbb{R}^n : c_1 x_1 + \cdots + c_n x_n \leq 1 \text{ for each } x \in S \}.$$ 

It is easy to check that $S^\circ$ is a closed convex subset of $\mathbb{R}^n$ that contains the origin, and we have $S \subseteq S^{\circ\circ}$, for every nonempty subset $S$ of $\mathbb{R}^n$. It is also plain that $S \subseteq T$ implies $S^\circ \supseteq T^\circ$ for any two nonempty subsets of $S$ and $T$ of $\mathbb{R}^n$.

Now, let $\mathcal{X}$ be the collection of all closed and convex subsets of $\mathbb{R}^n$ that contain the origin, and define the maps $f : 2^{\mathbb{R}^n} \to \mathcal{X}$ and $F : \mathcal{X} \to 2^{\mathbb{R}^n}$ as

$$f(S) := S^\circ \quad \text{and} \quad F(Y) := Y^\circ.$$ 

Then, $(f, F)$ is a Galois connection between $(2^{\mathbb{R}^n}, \supseteq)$ and $(\mathcal{X}, \subseteq)$. The argument is analogous to the one we gave in the previous example.

We conclude with an example from order theory. We shall later make good use of this example.

**Example 2.1.8.** Let $(X, \succ)$ be a poset, and define the self-maps $f$ and $F$ on $2^X$ by

$$f(S) := \{ x \in X : S \ni x \} \quad \text{and} \quad F(T) := \{ x \in X : x \ni T \}.$$ 

(That is, $f(S)$ is the collection of all $\ni$-lower bounds for $S$, and $F(T)$ is that of all $\ni$-upper bounds for $T$. Of course, $f(\emptyset) = X = F(\emptyset)$.) Then, $(f, F)$ is a Galois connection between $(2^X, \supseteq)$ and $(2^X, \subseteq)$. Indeed, for any nonempty subsets $S$ and $T$ of $X$,

$$S \ni y \text{ for every } y \in T \quad \text{iff} \quad x \ni T \text{ for every } x \in S.$$ 

It follows that $T \subseteq f(S)$ iff $F(T) \supseteq S$ for every $S$ and $T$ in $2^X$.
2.2 Properties of Galois Connections

The following characterization of Galois connections are occasionally useful.

**Proposition 2.2.1.** Let \((X, \preceq_X)\) and \((Y, \preceq_Y)\) be two preordered sets, and \(f : X \to Y\) and \(F : Y \to X\) two functions. Then, \((f, F)\) is a Galois connection between \((X, \preceq_X)\) and \((Y, \preceq_Y)\) if, and only if, both \(f\) and \(F\) are order-preserving, and

\[
F(f(x)) \preceq_X x \quad \text{and} \quad y \preceq_Y f(F(y))
\]

for every \(x\) in \(X\) and \(y\) in \(Y\).

**Proof.** Let \((f, F)\) be a Galois connection between \((X, \preceq_X)\) and \((Y, \preceq_Y)\). Take any \(x\) in \(X\), and notice that \(f(x) \preceq_Y f(x)\) because \(\preceq_Y\) is reflexive. Then, as \((f, F)\) is a Galois connection, \(F(f(x)) \preceq_X x\). In turn, if \(x\) and \(z\) are two elements of \(X\) such that \(x \preceq_X z\), then \(F(f(x)) \preceq_X z\) by (2) and transitivity of \(\preceq_X\). We thus find \(f(x) \preceq_Y f(z)\) because \((f, F)\) is a Galois connection. Conclusion: \(f\) is order-preserving and \(F(f(x)) \preceq_X x\) for every \(x \in X\). That \(F\) is order-preserving and the second part of (2) follow from this fact via the duality principle for Galois connections.

Conversely, assume that \(f\) and \(F\) are order-preserving and (2) is valid for every \(x\) in \(X\) and \(y\) in \(Y\). Then, for an arbitrarily fixed \((x, y) \in X \times Y\) with \(y \preceq_Y f(x)\), we have

\[
F(y) \preceq_X F(f(x)) \preceq_X y
\]

because \(F\) is order-preserving and (2) holds. By transitivity of \(\preceq_X\), therefore, \(F(y) \preceq_X x\). That \(F(y) \preceq_X x\) implies \(y \preceq_Y f(x)\) is similarly proved.

**Note.** In fact, this alternative way of looking at Galois connections is how Ore defined the concept originally. The equivalence of the definition we gave here and Ore’s definition was noticed first by Schmidt (1953).

While being a left adjoint is weaker than being an order-isomorphism, Proposition 2.2.1 shows that it is stronger than being order-preserving:

\[
\text{order-isomorphism} \implies \text{lower adjoint} \implies \text{order-preserving map}
\]

The converse implication is false, that is, an order-preserving map from one preordered set to another need not be a lower adjoint between these preordered sets. For instance, the self-map we considered in Example 1.2.1 is order-preserving but it is not a lower adjoint. (It is not difficult to prove this from scratch, but we will shortly be able to see this as a triviality.)
In what follows, we focus on the more special case of Galois connections between two posets. An important observation about such connections is the following:

**Proposition 2.2.2.** Let \((f, F)\) be a Galois connection between the posets \((X, \succ_X)\) and \((Y, \succ_Y)\). Then,

\[
f \circ F \circ f = f \quad \text{and} \quad F \circ f \circ F = F.
\]

**Proof.** Take any \(x\) in \(X\). By Proposition 2.2.1, we have \(F(f(x)) \succ_X x\) and hence, as \(f\) is order-preserving,

\[
f(F(f(x))) \succ_Y f(x).
\]

On the other hand, by reflexivity of \(\succ_X\), we have \(F(f(x)) \succ_X F(f(x))\), and hence, as \((f, F)\) is a Galois connection,

\[
f(x) \succ_Y f(F(f(x))).
\]

As \(\succ_Y\) is antisymmetric, therefore, \(f(F(f(x))) = f(x)\), establishing the first part of our assertion. The second part of this assertion follows from its first part and the duality principle for Galois connections.

Proposition 2.2.2 gives another perspective to the comparison of order-isomorphisms and lower adjoints. Obviously, if a map \(f\) is an order-isomorphism between two posets, then the inverse of \(f\) maps the image of any element in the domain back to that element. Put differently, in this case we have two maps, \(f\) and \(f^{-1}\), such that \(f\) transforms any one element of the domain to an element of the range and \(f^{-1}\) transforms that element back to where we started. Proposition 2.2.2 shows that something similar (but much weaker) holds when \(f\) is instead a lower adjoint. Apparently, in that case we have two maps, \(f\) and the upper adjoint \(F\), such that \(f\) transforms any one element \(x\) of the domain to an element \(y\) of the range and \(F\) transforms \(y\) back to another element \(z\) in the domain. It may be that \(z\) is distinct from \(x\), but it must be the case that \(f\) would transform \(z\) exactly where it maps \(x\), that is, we have \(f(x) = f(z)\). Put differently, if we agree to call two elements of \(X\) as \(f\)-equivalent when their images under \(f\) are the same, we see that \(x\) is \(f\)-equivalent to \(F(f(x))\) for every \(x\) in \(X\). (Obviously, were \(f\) an order-isomorphism, we would have the \(f\)-equivalence relation would coincide with \(\triangle_X\), that is, \(x = F(f(x))\) for every \(x\) in \(X\), which is the same thing as saying that \(F = f^{-1}\).)

As an immediate corollary of Proposition 2.2.2, we see now that while a lower adjoint may fail to be an order-isomorphism, it nevertheless acts as an order-isomorphism between a restriction of its domain and its (unrestricted) range. That is:
**Corollary 2.2.3.** Let \((f, F)\) be a Galois connection between the posets \((X, \geq_X)\) and \((Y, \geq_Y)\). Then, \(f|_{F(Y)}\) is an order-isomorphism between \((F(Y), \geq_X)\) and \((f(X), \geq_Y)\).

Here is a quick application of Proposition 2.2.2.

**Example 2.2.1.** For any open subset \(O\) of a topological space \(X\), we have
\[
\text{cl} (\text{int} (\text{cl}(O))) = \text{cl}(O).
\]
On the other hand, in the context of any inner product space \((X, \langle \cdot, \cdot \rangle)\), we have
\[
S^{\perp\perp} = S^\perp
\]
for every subset \(S\) of \(X\). Similarly,
\[
S^{````} = S^{``}
\]
for every subset of \(\mathbb{R}^n\). These seemingly disparate results are unified under the theory of Galois connections. They are established at one stroke by applying Proposition 2.2.2 to Examples 2.1.5, 2.1.6 and 2.1.7.

By definition, a Galois connection determines a lower adjoint. We have so far said nothing about the converse of this fact (although we have implicitly acted as if the converse is true). Our next result shows that, in the case of posets, every lower adjoint is compatible with exactly one upper adjoint. Furthermore, it gives a precise formula for this upper adjoint.

**Proposition 2.2.4.** Let \((f, F)\) be a Galois connection between the preordered sets \((X, \succ_X)\) and \((Y, \succ_Y)\). Then,
\[
F(y) \in \max\{x \in X : y \succ_Y f(x)\,\succ_X\}
\]
for every \(y\) in \(Y\).

**Proof.** Take any \(y\) in \(Y\), and define \(S := \{x \in X : y \succ_Y f(x)\}\). As \((f, F)\) is a Galois connection between \((X, \succ_X)\) and \((Y, \succ_Y)\), we have \(F(y) \succ_X x\) for every \(x \in S\), that is, \(F(y) \succ_X S\). Moreover, by Proposition 2.2.1, \(y \succ_Y f(F(y))\), and hence \(F(y) \in S\). It then follows that \(F(y)\) is a \(\succ_X\)-maximum element in \(S\).

The following is an immediate consequence of Proposition 2.2.4.

**Corollary 2.2.5.** Let \((f, F)\) be a Galois connection between the posets \((X, \succ_X)\) and \((Y, \succ_Y)\). Then,
\[
F(y) = \text{the } \succ_X\text{-maximum element of } \{x \in X : y \succ_Y f(x)\}
\]
for every \(y\) in \(Y\).
2.3 $\lor$-Preservation

Our discussion so far does not make transparent why two Galois connected posets are in fact “closely related.” This is obvious in the case of order-isomorphic posets, as any two such posets have identical order-theoretic properties. For instance, if one has a maximum and/or minimum element, so must the other. More generally, an order-isomorphism preserves suprema and infima (when they exist). We shall now show that Galois connected posets also share certain important order-theoretic properties. Indeed, it turns out that a lower adjoint surely preserves suprema (but not necessarily infima). This is the most important property of lower adjoints.

**Proposition 2.3.1.** Let $(f, F)$ be a Galois connection between the posets $(X, \geq_X)$ and $(Y, \geq_Y)$. Then, $f$ is $\lor$-preserving, that is,

$$f(\lor S) = \lor f(S)$$

for every subset $S$ of $X$ such that $\lor S$ exists. Similarly, $F$ is $\land$-preserving.

**Proof.** Let $S$ be a subset of $X$ such that $\lor S \in X$. By Proposition 2.2.1, $f$ is order-preserving. As $\lor S \geq_X S$, therefore, $f(\lor S) \geq_Y f(S)$, that is, $f(\lor S)$ is an $\geq_Y$-upper bound for $f(S)$. Suppose $y$ is another $\geq_Y$-upper bound for $f(S)$, that is, $y \geq_Y f(S)$. Then, as $(f, F)$ is a Galois connection, $F(y) \geq_X S$. Therefore, $F(y) \geq_X \lor S$, and hence, $y \geq_Y f(\lor S)$ because $(f, F)$ is a Galois connection. It follows that $f(\lor S)$ is the the $\geq_X$-supremum of $f(S)$, as we sought. Our second assertion follows from the first via the duality principle for Galois connections.

This result sharpens our understanding of where being a lower adjoint is located in the hierarchy of monotonicity properties. Apparently:

| order-isomorphism | $\Rightarrow$ | lower adjoint | $\Rightarrow$ | $\lor$-preserving map | $\Rightarrow$ | order-preserving map |

Looking back at Example 1.2.1 now shows that the map $f$ considered there is order-preserving, but as it does not preserve suprema, it is not a lower adjoint. (As we promised, this observation is now a triviality.) It is a bit harder to see that there are $\lor$-preserving maps that are not lower adjoints. We leave providing an example to this effect as an exercise.

While being $\lor$-preserving is in general less demanding than being a lower adjoint, it is remarkable that these two properties coincide for those maps whose domains are complete lattices. This is the main result of this section (whose proof is already suggested by Corollary 2.2.5).
Theorem 2.3.2. Let \((X, \succeq_X)\) be a complete lattice and \((Y, \succeq_Y)\) a poset. Then, a function \(f : X \to Y\) is \(\lor\)-preserving if, and only if, it is a lower adjoint between \((X, \succeq_X)\) and \((Y, \succeq_Y)\).

Proof. In view of Proposition 2.3.1, we only need to establish the “only if” part of this assertion. To this end, assume that \(f\) is \(\lor\)-preserving, and define \(F : Y \to X\) by

\[ F(y) = \bigvee \{ \omega \in X : y \succeq_Y f(\omega) \}. \]

\(F\) is well-defined because \((X, \succeq_X)\) is a complete lattice. It is order-preserving because \(\succeq_Y\) is transitive, while \(f\) is order-preserving because every \(\lor\)-preserving map is. On the other hand, for any \(x\) in \(X\),

\[ F(f(x)) = \bigvee \{ \omega \in X : f(x) \succeq_Y f(\omega) \} \succeq_X x \]

because \(\succeq_Y\) is reflexive. Finally, take an arbitrary \(y\) in \(Y\), and set \(S := \{ \omega \in X : y \succeq_Y f(\omega) \}\). Then, \(y \succeq_Y f(S)\), and hence, as \(f\) is \(\lor\)-preserving,

\[ y \succeq_Y \bigvee f(S) = f(\bigvee S) = f(F(y)). \]

We may now invoke Proposition 2.2.1 to conclude that \((f, F)\) is a Galois connection between \((X, \succeq_X)\) and \((Y, \succeq_Y)\).

Thanks to Theorem 2.3.2, to see if a given map from a complete lattice into a poset is \(\lor\)-preserving, we may check if that map is a lower adjoint between the involved posets. This is a useful strategy because there is a straightforward method for checking if a given map is a part of a Galois connection as its lower adjoint. After all, there is a unique upper adjoint for any lower adjoint. Indeed, combining Theorem 2.3.2 and Corollary 2.2.5 yields:

Corollary 2.3.3. Let \((X, \succeq_X)\) be a complete lattice and \((Y, \succeq_Y)\) a poset. Then, a function \(f : X \to Y\) is \(\lor\)-preserving if, and only if, \((f, F)\) is a Galois connection between \((X, \succeq_X)\) and \((Y, \succeq_Y)\), where

\[ F(y) = \text{the } \succeq_X\text{-maximum element of } \{ x \in X : y \succeq_Y f(x) \} \]

for every \(y\) in \(Y\).

We conclude with a simple application of Proposition 2.3.1.

Example 2.3.1. For any collection \(\mathcal{O}\) of open subsets of a topological space \(X\), we have

\[ \text{cl}(\bigcup \mathcal{O}) = \bigcup \{ \text{cl}(O) : O \in \mathcal{O} \}. \]
On the other hand, in the context of any inner product space \((X, \langle \cdot, \cdot \rangle)\), we have
\[
(\bigcup S)^\perp = \bigcap \{S^\perp : S \in S\}
\]
for every subset \(S\) of \(2^X\). Similarly,
\[
(\bigcup S)^\circ = \bigcap \{S^\circ : S \in S\}.
\]
for every collection \(S\) of subsets of \(\mathbb{R}^n\). We see here again the unifying power of the theory of Galois connections. All of these results are obtained readily by applying Proposition 2.3.1 to Examples 2.1.5, 2.1.6 and 2.1.7.

2.4 Application: The Dedekind-MacNeille Completion

Let \(X\) be a nonempty set. By definition, a \(\supseteq\)-closure operator \(c\) on \(2^X\) is a \(\supseteq\)-preserving self-map on \(2^X\) such that
\[
c(S) \supseteq S \quad \text{and} \quad c(c(S)) = c(S)
\]
for every subset \(S\) of \(X\). An important observation is that the range \(c(2^X)\) of such a self-map \(c\) is a complete lattice (relative to \(\supseteq\)). Furthermore, for any subset \(S\) of \(c(2^X)\), we have
\[
\text{the } \supseteq\text{-supremum of } S \text{ in } c(2^X) = c(\bigcup S)
\]
and
\[
\text{the } \supseteq\text{-infimum of } S \text{ in } c(2^X) = \bigcap S.
\]
You were asked to establish even more general facts than these in Exercise 8.14 of Chapter 1.

A natural way of specifying a \(\supseteq\)-closure operator on \(2^X\) is, in fact, by using a suitable Galois connection.

**Lemma 2.4.1.** Let \(X\) be a nonempty set and \((f, F)\) a Galois connection between \((2^X, \supseteq)\) and a poset \((Y, \sqsubseteq)\). Then, \(F \circ f\) is a \(\supseteq\)-closure operator on \(2^X\).

**Proof.** By Proposition 2.2.1, \(f\) and \(F\) are order-preserving, and hence \(F \circ f\) is a \(\supseteq\)-preserving self-map on \(2^X\), while, by the same result, we have \(F \circ f(S) \supseteq S\) for every subset \(S\) of \(X\). Finally, by Proposition 2.2.2,
\[
(F \circ f) \circ (F \circ f) = F \circ (f \circ F \circ f) = F \circ f,
\]
and hence the lemma.
We shall now use this simple consequence of the theory of Galois connections to establish a famous result of order theory. Let \((X, \succsim)\) be a poset. Define the self-maps \(f\) and \(F\) on \(2^X\) by

\[
f(S) := \text{the collection of all } \succsim\text{-upper bounds for } S, \tag{4}
\]

and

\[
F(T) := \text{the collection of all } \succsim\text{-lower bounds for } T. \tag{5}
\]

In Example 2.1.8, we have observed that \((f, F)\) is a Galois connection between \((2^X, \supseteq)\) and \((2^X, \subseteq)\), so by Lemma 2.4.1, \(F \circ f\) is a \(\supseteq\)-closure operator on \(2^X\).

Next, consider the function \(\psi : X \to 2^X\) defined by

\[
\psi(x) := x^\perp.
\]

This map has a number of interesting properties. First, for every \(x \in X\), we have

\[
F(f(\{x\})) = F(x^\perp) = x^\perp,
\]

which means that \(\psi(X)\) is a subset of the range of \(F \circ f\). Therefore, \(\psi\) is an order-embedding from \((X, \succsim)\) into \((F \circ f(2^X), \supseteq)\). Second, \(\psi\) is both \(\sqcup\)-preserving and \(\sqcap\)-preserving. To see this, take any subset \(S\) of \(X\) with \(\sqcup S \in X\). First, note that

\[
\psi(\sqcup S) = (\sqcup S)^\perp \\
= F(\{\sqcup S\}) \\
= F \circ f \circ F(\{\sqcup S\})
\]

by Proposition 2.2.2. Second, in view of (3), where \(F \circ f\) acts as the order-closure operator \(\xi\),

\[
\sqcup \psi(S) = F \circ f(\bigcup \{x^\perp : x \in S\}) \\
= F \circ f((\sqcup S)^\perp) \\
= F \circ f \circ F(\{\sqcup S\}),
\]

where the second equality follows from the fact that the \(\succsim\)-upper bounds for \((\sqcup S)^\perp\) and \(\bigcup \{x^\perp : x \in S\}\), that is, the images of these sets under \(f\), coincide. We leave the proof for the \(\sqcap\)-preservation property of \(\psi\) (which is easier) as an exercise.

Finally, we note that \((F \circ f(2^X), \supseteq)\) is a complete lattice – it is called the Dedekind-MacNeille completion of \((X, \succsim)\) – simply because the range of every \(\supseteq\)-closure operator on \(2^X\) is a complete lattice.
Note. The Dedekind-MacNeille completion of an arbitrary poset was introduced by Holbrook MacNeille in 1937. As the construction of this completion is clearly inspired by how \( \mathbb{Q} \) is extended to \( \mathbb{R} \) by means of Dedekind cuts, Richard Dedekind’s name is also associated with it.

In summary:

**Theorem 2.4.2.** Every poset can be order-embedded in a complete lattice by an order-embedding that is both \( \lor \)-preserving and \( \land \)-preserving.

Here are a few illustrations.

**Example 2.4.1.** Let \( X \) be a nonempty set, and define the maps \( f \) and \( F \) by (4) and (5), with \( \supseteq \) being \( \supseteq X \). Then, for any subset \( S \) of \( X \),

\[
F \circ f(S) = \begin{cases} \ X, & \text{if } |S| > 1, \\ \ S, & \text{if } |S| = 1, \\ \emptyset, & \text{if } |S| = 0. \end{cases}
\]

Therefore, the Dedekind-MacNeille completion of \( (X, \supseteq X) \) is — “is” in the sense of “is order-isomorphic to” — the complete lattice \( (X \cup \{x_*, x^*\}, \supseteq) \) where \( x_* \) and \( x^* \) are two objects that do not belong to \( X \), and \( \supseteq \) is the partial order on \( X \cup \{x_*, x^*\} \) with \( x^* \supseteq X \supseteq x_* \) and \( \supseteq \cap (X \times X) = \Delta_X \).

**Example 2.4.2.** The Dedekind-MacNeille completion of the loset \( (\mathbb{N}, \supseteq) \) “is” \( (\{1, 2, \ldots, \infty\}, \supseteq) \). Indeed, for any set \( S \) of positive integers, we have

\[
F \circ f(S) = \begin{cases} \ \{\max S\}, & \text{if } \max S \neq \emptyset, \\ \ \{1\}, & \text{if } S = \emptyset, \\ \ \mathbb{N}, & \text{otherwise,} \end{cases}
\]

where \( f \) and \( F \) are defined by (4) and (5), with \( \supseteq \) being the usual order \( \supseteq \) of \( \mathbb{N} \). Therefore, the Dedekind-MacNeille completion of \( (\mathbb{N}, \supseteq) \) is

\( (\{\{1\}, \{1, 2\}, \ldots, \{\mathbb{N}\}\}, \supseteq) \),

and obviously, this complete lattice is order-isomorphic to \( (\{1, 2, \ldots, \infty\}, \supseteq) \).

**Example 2.4.3.** The Dedekind-MacNeille completion of the loset \( (\mathbb{Q}, \supseteq) \) “is” the loset \( (\mathbb{R}, \supseteq) \), where \([\overline{-\infty, \infty}], \supseteq) \) is the set of all extended real numbers and \( \supseteq \) the obvious extension of the usual order on \( \mathbb{R} \). Clearly, for any set \( S \) of rational numbers, we have

\[
F \circ f(S) = \begin{cases} \ \{r \in \mathbb{Q} : r \leq \sup S\}, & \text{if } \sup S \in \mathbb{R}, \\ \ \emptyset, & \text{if } S = \emptyset, \\ \ \mathbb{Q}, & \text{otherwise,} \end{cases}
\]
where $f$ and $F$ are defined by (4) and (5), with $\succ$ being $\geq$. Therefore, the Dedekind-MacNeille completion of $(\mathbb{N}, \geq)$ is

$$\{\emptyset\} \cup \{(-\infty, x] \cap \mathbb{Q} : x \in \mathbb{R}\} \cup \{\mathbb{Q}, \geq\},$$

and obviously, this complete lattice is order-isomorphic to $([-\infty, \infty], \geq)$.

Needless to say, there is nothing special about order-embedding a given poset $(X, \succ)$ in a complete lattice (Proposition 1.3.1). What makes the Dedekind-MacNeille completion of a poset special is that this completion departs from the original poset in a minimal way. For example, it follows from Theorem 2.4.2 that the Dedekind-MacNeille completion of a complete lattice is order-isomorphic to itself. More generally:

**Proposition 2.4.3.** Let $(X, \succ)$ be a poset, and $(Y, \succeq)$ a complete lattice into which $(X, \succ)$ can be order-embedded. Then, the Dedekind-MacNeille completion of $(X, \succ)$ can also be order-embedded in $(Y, \succeq)$.

This result, whose proof is left as an exercise, provides a sense in which we can think of the Dedekind-MacNeille completion of a poset as the “smallest possible” complete lattice into which we can order-embed that poset. Another perspective on this matter is provided in Exercise 2.11.

**Exercises**

2.1. Let $X$ be a set and $Y$ a nonempty subset of $X$. Define the self-maps $f$ and $F$ on $2^X$ by

$$f(S) := S \cap Y \quad \text{and} \quad F(T) := T \cup (X \setminus Y).$$

Show that $(f, F)$ is a Galois connection between $(2^X, \supseteq)$ and itself.

2.2. Let $X$ be a topological space, and $\varphi$ a continuous self-map on $X$. Define the self-maps $f$ and $F$ on $C_X$ by

$$f(S) := \text{cl}(\varphi(S)) \quad \text{and} \quad F(C) := \varphi^{-1}(C).$$

Show that $(f, F)$ is a Galois connection between $(C_X, \supseteq)$ and itself.

2.3. (Generalization of Example 2.1.6) Let $X$ be a linear space, and denote the collection of all linear functionals on $X$ by $X^*$. Recall that the annihilator of a subset $S$ of $X$ is defined as

$$S^\perp := \{L \in X^* : L(x) = 0 \text{ for each } x \in S\}.$$

In turn, for any linear subspace $\mathcal{L}$ of $X^*$, we define

$$\mathcal{L}^\perp := \{x \in X : L(x) = 0 \text{ for each } L \in \mathcal{L}\}.$$
Now let \( \mathcal{X} \) stand for the collection of all linear subspaces of \( X^* \), and define the maps \( f : 2^X \to \mathcal{X} \) and \( F : \mathcal{X} \to 2^X \) by \( f(S) := S^\circ \) and \( F(\mathcal{L}) := \mathcal{L}^\circ \). Prove: \( (f, F) \) is a Galois connection between \((2^X, \supseteq)\) and \((\mathcal{X}, \subseteq)\).

2.4. Let \((X, \supseteq)\) be a lattice and \( Y \) a subset of \( X \) such that for every \( x \) in \( X \) there is a subset \( S \) of \( Y \) with \( x = \sqrt{S} \). (Such a set \( Y \) is said to be \( \vee \)-dense in \( X \), and \( \wedge \)-dense subsets of \( X \) are dually defined.) Let \( S := \{ y^1 \cap Y : y \in Y \} \), and define \( f : S \to X \) by \( f(S) := \sqrt{S} \). Prove that \( f \) is a lower adjoint between \((S, \supseteq)\) and \((X, \supseteq)\).

2.5. Prove: If \((f, F)\) is a Galois connection between the preordered sets \((X, \supseteq_X)\) and \((Y, \supseteq_Y)\), and \((g, G)\) a Galois connection between \((Y, \supseteq_Y)\) and \((Z, \supseteq_Z)\), then \((g \circ f, F \circ G)\) is a Galois connection between \((X, \supseteq_X)\) and \((Z, \supseteq_Z)\).

2.6. Let \((f, F)\) be a Galois connection between the posets \((X, \supseteq_X)\) and \((Y, \supseteq_Y)\). Prove that \( f \) is injective iff \( F \) is surjective.

2.7. Give an example of a \( \vee \)-preserving map between two posets which is not a lower adjoint between those posets.

2.8. (Pataki Connections) Let \((X, \supseteq_X)\) and \((Y, \supseteq_Y)\) be two posets, and \( f : X \to Y \) and \( \varphi : X \to X \) two functions. The pair \((f, \varphi)\) is called a Pataki connection between \((X, \supseteq_X)\) and \((Y, \supseteq_Y)\) if

\[
\varphi(x) \supseteq_Y f(y) \text{ iff } \varphi(x) \supseteq_X y
\]

for every \( x, y \in X \). Prove:

- If \((f, F)\) is a Galois connection between \((X, \supseteq_X)\) and \((Y, \supseteq_Y)\), then \((f, F \circ f)\) is a Pataki connection between \((X, \supseteq_X)\) and \((Y, \supseteq_Y)\).

- If \((f, \varphi)\) is a Pataki connection between \((X, \supseteq_X)\) and \((Y, \supseteq_Y)\), then \( f = f \circ \varphi \), and \( \varphi(x) \supseteq_X x \) for each \( x \in X \). Is the converse true?

- \( \epsilon \) is a \( \supseteq_X \)-closure operator on \( X \) – recall Exercise 8.14 of Chapter 1 – iff \((\epsilon, \epsilon)\) is a Pataki connection between \((X, \supseteq_X)\) and \((X, \supseteq_X)\).

- \((f, \varphi)\) is a Pataki connection between \((X, \supseteq_X)\) and \((Y, \supseteq_Y)\) iff \( \varphi \) is a \( \supseteq_X \)-closure operator on \( X \) and \( f(x) \supseteq_Y f(y) \text{ iff } \varphi(x) \supseteq_X \varphi(y) \), for every \( x, y \in X \).

2.9. Let \( X \) be a nonempty set, and \( \mathcal{X} \) the collection of all subsets of \( X \) that are of the form either \( \{x\} \) or \( X \setminus \{x\} \) for some \( x \) in \( X \). Find the Dedekind-MacNeille completion of \((\mathcal{X}, \supseteq)\).

2.10. Let \( X \) be a nonempty set, and define \( \mathcal{X} := \{ S \in 2^X : \min\{|S|, |X \setminus S| \} < \infty \} \). We have seen in Example 8.1.5 of Chapter 1 that \((\mathcal{X}, \supseteq)\) is an incomplete lattice. Find the Dedekind-MacNeille completion of \((\mathcal{X}, \supseteq)\).

2.11. Prove Proposition 2.4.3.

2.12. Let \((X, \supseteq)\) be a poset and set \( \mathcal{X} := F \circ f(2^X) \), where \( f \) and \( F \) are defined by (4) and (5). Denoting the map \( x \mapsto x^\dagger \) on \( X \) by \( \psi \), show that \((\psi(X), \supseteq)\) is both \( \vee \)-dense and \( \wedge \)-dense in \( \mathcal{X} \) (relative to \( \supseteq)\).

2.13. Let \((X, \supseteq)\) be a poset and \( Y \) a nonempty subset of \( X \). Let \((\mathcal{X}, \supseteq)\) and \((Y, \supseteq)\) be the Dedekind-MacNeille completions of \((X, \supseteq)\) and \((Y, \supseteq)\), respectively. Prove that \((Y, \supseteq)\) can be order-embedded in \((X, \supseteq)\).
3 Order-Preserving Correspondences

3.1 Induced Set-Orderings

There are various interesting ways in which we can order the nonempty subsets of a given set by using a partial order defined on that set. We shall consider two ways of doing this here.

**Definition.** Let \((X, \preceq)\) be a preordered set. The **lower set-ordering** induced by \(\preceq\) is the binary relation \(\preceq\) on \(2^X \setminus \{\emptyset\}\) such that \(A \preceq B\) if and only if \(B \cap x^\uparrow \neq \emptyset\) for every \(x \in A\). Similarly, the **upper set-ordering** induced by \(\preceq\) is the binary relation \(\preceq^\bullet\) on \(2^X \setminus \{\emptyset\}\) such that \(A \preceq^\bullet B\) if for every \(y\) in \(B\) there is a \(z\) in \(A\) with \(z \preceq y\). In turn, \(\preceq^w := \preceq \cap \preceq^\bullet\) is called the **weak set-ordering** induced by \(\preceq\), that is, \(\preceq^w\) is the binary relation \(\preceq^w\) on \(2^X \setminus \{\emptyset\}\) such that \(A \preceq^w B\) if for every \((x, y) \in A \times B\),

\[
x \preceq^w y \text{ for some } w \in B \quad \text{and} \quad z \preceq y \text{ for some } z \in A.
\]

Given any preordered set \((X, \preceq)\), the relations \(\preceq\), \(\preceq^\bullet\) and \(\preceq^w\) are loyal to \(\preceq\) in the sense that they agree with \(\preceq\) on the ranking of singleton sets, that is,

\[
\{x\} \preceq^w \{y\} \iff x \preceq y
\]

for every \(x, y \in X\). Furthermore, any one of these relations is a preorder.

**Proposition 3.1.1.** If \((X, \preceq)\) is a preordered set, then both \(\preceq\) and \(\preceq^\bullet\) are preorders on \(2^X \setminus \{\emptyset\}\). Thus: \((2^X \setminus \{\emptyset\}, \preceq^w)\) is a preordered set.

The proof is a straightforward exercise, but note that the antisymmetry of \(\preceq\) would not guarantee that of \(\preceq^w\), that is, the weak set-ordering induced by a partial order need not be a partial order. (For instance, we have \([0, 1] \preceq^w \{0, 1\}\) and \(\{0, 1\} \preceq^w [0, 1]\) simultaneously.) Having said this, we should also note that two sets \(A\) and \(B\) that are rendered “equivalent” by the weak set-ordering must have the same extremal elements. That is, as one can easily verify,

\[
A \preceq^w B \preceq^w A \quad \text{implies} \quad \text{MAX}(A, \preceq) = \text{MAX}(B, \preceq),
\]

and similarly for \(\preceq\)-minimal elements in \(A\) and \(B\). In the case of posets, the same also goes for suprema and infima. More generally:

**Proposition 3.1.2.** Let \((X, \succ)\) be a poset. Then, for any nonempty subsets \(A\) and \(B\) of \(X\),

\[
A \succ B \quad \text{implies} \quad \bigwedge A \succeq \bigwedge B,
\]
provided that $\bigwedge A$ and $\bigwedge B$ exist, and

$$A \succ^* B \quad \text{implies} \quad \bigvee A \succ \bigvee B,$$

provided that $\bigvee A$, and $\bigvee B$ exist.

**Proof.** Take any two nonempty subsets $A$ and $B$ of $X$ such that $\bigwedge A$ and $\bigwedge B$ exist. Assume $A \succ^* B$, and consider any element $x$ of $A$. By definition of $\succ^*$, there is an element $w$ of $B$ such that $x \succ w$. It follows that $x \succ \bigwedge B$, that is, $\bigwedge B$ is a $\succ$-lower bound for $A$, and hence, $\bigwedge A \succ \bigwedge B$, as we sought. The second assertion is proved analogously.

When $(X, \succ)$ is a lattice, we have another useful way of inducing a set ordering on $2^X \setminus \{\emptyset\}$.

**Definition.** Let $(X, \succ)$ be a lattice. The **strong set-ordering** induced by $\succ$ is the binary relation $\succ^s$ on $2^X \setminus \{\emptyset\}$ such that $A \succ^s B$ iff for every $(x, y) \in A \times B$,

$$x \lor y \in A \quad \text{and} \quad x \land y \in B.$$

Given any lattice $(X, \succ)$, the strong set-ordering induced by $\succ$ is loyal to the partial order $\succ$ just like the weak set-ordering. That is,

$$\{x\} \succ^s \{y\} \quad \text{iff} \quad x \succ y$$

for every $x, y \in X$. Furthermore, the strong set-ordering is quite close to being a partial order.

**Proposition 3.1.3.** If $(X, \succ)$ is a lattice, then $\succ^s$ is antisymmetric and transitive.

We leave the proof of this observation as an exercise, but note that, in general, $\succ^s$ need not be reflexive. For instance, if $S$ is the subset of $\mathbb{R}^2$ that consists of $(1, 0)$ and $(0, 1)$, then $S \succ^s S$ is false. But, of course, this difficulty disappears when ranking the sublattices of the original lattice. That is:

**Corollary 3.1.4.** Let $(X, \succ)$ be a lattice, and $S$ the collection of all subsets $S$ of $X$ such that $(S, \succ)$ is a sublattice of $(X, \succ)$. Then, $(S, \succ^s)$ is a poset.

Finally, a comparison of the weak and strong set-orderings is in order. Given a lattice $(X, \succ)$, as one might expect, $\succ^s$ is stronger than $\succ^w$ in the sense that

$$A \succ^s B \quad \text{implies} \quad A \succ^w B$$
for any nonempty subsets $A$ and $B$ of $X$. (Indeed, if $A \succ^s B$, then, for any $(x, y)$ in $A \times B$, we have $z := x \lor y \in A$ and $w := x \land y \in B$, while we obviously have $x \succ w$ and $z \succ y$.) The converse is false. That is, $A \succ^w B$ may hold while $A \succ^s B$ does not. For instance, within the context of $(\mathbb{R}^2, \succeq)$, we have

$$\{(1, 0), (0, 1), (2, 2)\} \succ^w \{(1, 0), (0, 1)\},$$

while $\succ^w$ cannot be replaced here with $\succeq^s$ because $(1, 0) \lor (0, 1)$ does not belong to the former set.

We conclude by noting the following corollary of Proposition 3.1.2 and the fact that $\succ^s$ is stronger than $\succ^w$.

**Corollary 3.1.5.** Let $(X, \succ)$ be a lattice. Then, for any nonempty subsets $A$ and $B$,

$$A \succ^s B \quad \text{implies} \quad \lor A \succ \lor B,$$

provided that $\lor A$ and $\lor B$ exist.

### 3.2 Order-Preserving Correspondences

By a correspondence $\Gamma$ from a nonempty set $X$ into another nonempty set $Y$, we mean a map from $X$ into $2^Y \setminus \{\emptyset\}$. Thus, for each $x \in X$, the image $\Gamma(x)$ of $x$ under $\Gamma$ is a nonempty subset of $Y$. We write

$$\Gamma : X \rightrightarrows Y$$

to denote that $\Gamma$ is a correspondence from $X$ into $Y$. Just as in the case of functions, $X$ is called the **domain** of $\Gamma$, and $Y$ the **codomain** of $\Gamma$. For any nonempty subset $S$ of $X$, we let

$$\Gamma(S) := \bigcup \{\Gamma(x) : x \in S\},$$

and by convention, set $\Gamma(\emptyset) := \emptyset$. The set $\Gamma(X)$ is called the **range** of $\Gamma$. If $\Gamma(X) \subseteq X$, we refer to $\Gamma$ as a **self-correspondence on** $X$.

Of course, every function $f : X \to Y$ can be viewed as a particular correspondence from $X$ into $Y$. Indeed, there is no difference between $f$ and the correspondence $\Gamma : X \rightrightarrows Y$ defined by $\Gamma(x) := \{f(x)\}$ (other than the formalism of using $\{\cdot\}$ in denoting the images under $\Gamma$). Conversely, if $\Gamma$ is **single-valued**, that is, $|\Gamma(x)| = 1$ for every $x \in X$, then it can be thought of as a function mapping $X$ into $Y$. It is thus customary to identify the terms “single-valued correspondence” and “function,” and we shall do so here as well.
**Note.** For any nonempty set $X$, there is a tight connection between the binary relations on $X$ and self-correspondences on $X$. First, every reflexive binary relation $R$ on $X$ induces a self-correspondence $\Gamma_R$ on $X$ by

$$\Gamma_R(x) := \{ y \in X : y \, R \, x \}.$$  

Second, if $\Gamma$ is a self-correspondence on $X$, then

$$\{(x, y) \in X \times X : y \in \Gamma(x)\}$$

is a binary relation on $X$.

**Note.** The literature is not unified in the way it refers to correspondences. Some mathematicians call them *multifunctions*, some *many-valued maps*, and still others refer to them as *set-valued maps*.

The induced set orderings we discussed in the previous subsection allow us to extend the notion of “order-preservation” to the context of correspondences in a straightforward manner.

**Definition.** Let $(X, \succeq_X)$ and $(Y, \succeq_Y)$ be two preordered sets. A correspondence $\Gamma : X \rightarrow Y$ is said to be **lower order-preserving** if

$$x \succeq_X y \quad \text{implies} \quad \Gamma(x) \succeq_Y \Gamma(y),$$

and **upper order-preserving** if

$$x \succeq_X y \quad \text{implies} \quad \Gamma(x) \succeq_Y^\circ \Gamma(y)$$

for every $x$ and $y$ in $X$. If $\Gamma$ is both lower and upper order-preserving, we say that it is **weakly order-preserving**. If, in addition, $X = Y$ and $\succeq_X = \succeq_Y$, then we refer to $\Gamma$ as **weakly $\succeq$-preserving**.

Similarly:

**Definition.** Let $(X, \succeq_X)$ be a preordered set and $(Y, \succeq_Y)$ a lattice. A correspondence $\Gamma : X \rightarrow Y$ is said to be **strongly order-preserving** if

$$x \succeq_X y \quad \text{implies} \quad \Gamma(x) \succeq_Y^s \Gamma(y),$$

for every $x$ and $y$ in $X$. If $X = Y$ and $\succeq_X = \succeq_Y$ here, then we say that $\Gamma$ is **strongly $\succeq$-preserving**.

From the comparison of the weak and strong set-orderings it follows that every strongly order-preserving correspondence (whose codomain is a lattice) is weakly order-preserving, and not conversely.
As a final order of business here, we shall talk about order-preserving selections from a correspondence.

**Definition.** Let \((X, \preceq_X)\) and \((Y, \preceq_Y)\) be two preordered sets, and \(\Gamma : X \Rightarrow Y\) a correspondence. A **selection** from \(\Gamma\) is a function \(f : X \to Y\) such that \(f(x) \in \Gamma(x)\) for every \(x \in X\).

It is frequently useful to be able to find order-preserving selections from a given correspondence. The following two results provide some simple sufficient conditions for being able to do this.

**Proposition 3.2.1.** Let \((X, \preceq_X)\) be a preordered set, \((Y, \leq_Y)\) a poset, and \(\Gamma : X \Rightarrow Y\) a weakly order-preserving correspondence such that \(\max(\Gamma(x), \leq_Y) \neq \emptyset\) for every \(x \in X\). Then, there exists an order-preserving selection from \(\Gamma\).

**Proof.** We define \(f : X \to Y\) by

\[
f(x) := \text{the } \leq_Y \text{-maximum element of } \Gamma(x).
\]

By Proposition 3.1.2, \(f\) is an order-preserving selection from \(\Gamma\).

As the strong set-ordering is stronger than the weak set-ordering, the following is an immediate consequence of Proposition 3.2.1.

**Corollary 3.2.2.** Let \((X, \preceq_X)\) be a preordered set and \((Y, \succeq_Y)\) a lattice. Let \(\Gamma : X \Rightarrow Y\) be a strongly order-preserving correspondence such that \((\Gamma(x), \succeq_Y)\) is a subcomplete sublattice of \((Y, \succeq_Y)\) for every \(x \in X\). Then, there exists an order-preserving selection from \(\Gamma\).

**Exercises**

3.1. Let \(f\) be an increasing and surjective self-map on \(\mathbb{R}\), and define \(\Gamma : \mathbb{R} \Rightarrow \mathbb{R}\) by

\[
\Gamma(x) := \{\omega \in \mathbb{R} : f(\omega) = x\}.
\]

Is \(\Gamma\) weakly \(\succeq\)-preserving? Strongly \(\succeq\)-preserving?

3.2. Let \(f\) be an increasing real map on \(\mathbb{R}^n\), and define \(\Gamma : \mathbb{R} \Rightarrow \mathbb{R}^n\) by

\[
\Gamma(x) := \{\omega \in \mathbb{R}^n : f(\omega) \leq x\}.
\]

Is \(\Gamma\) weakly \(\succeq\)-preserving? Strongly \(\succeq\)-preserving?
3.3. Let \((X, \succ)\) be a lattice with no \(\succ\)-maximum and \(\succ\)-minimum elements. Define the self-correspondences \(\Gamma\) and \(\Upsilon\) by \(\Gamma(x) := x^\dagger\) and \(\Upsilon(x) := x^\dagger\), respectively. Are these correspondences weakly \(\succ\)-preserving? Strongly \(\succ\)-preserving?

3.4. Let \((X, \succ)\) be a lattice, and \(Y\) and \(Z\) two subsets of \(X\) such that \((Y, \succ)\) and \((Z, \succ)\) are sublattices of \((X, \succ)\). Assume that

\[
 Z \cap y^\dagger \cap x^\dagger \succ^\ast Y \cap y^\dagger \cap x^\dagger
\]

for every \(x\) and \(y\) in \(X\) such that both of these sets are nonempty. Show that \(Z \succ^\ast Y\).

3.5. Let \((X, \succ_X)\) be a finite poset and \((Y, \succ_Y)\) a lattice. Let \(\Gamma : X \Rightarrow Y\) be a strongly order-preserving correspondence such that \((\Gamma(x), \succ_Y)\) is a sublattice of \((Y, \succ_Y)\) for every \(x \in X\). Prove: If \(f\) is a selection from \(\Gamma\), then

\[
 x \mapsto \bigvee \{f(\omega) : x \succ_X \omega \in X\}
\]

is an order-preserving selection from \(\Gamma\).

3.6. Let \((X, \succ_X)\) be a poset and \((Y, \succ_Y)\) a lattice. Let \(\Gamma : X \Rightarrow Y\) be a strongly order-preserving correspondence such that \((\Gamma(x), \succ_Y)\) is a sublattice of \((Y, \succ_Y)\) for every \(x \in X\). True or false: \((\Gamma(X), \succ_Y)\) is a sublattice of \((Y, \succ_Y)\).

3.7. (Set-Valued Galois Connections) Let \((X, \succ_X)\) and \((Y, \succ_Y)\) be two posets and \(\Gamma_\ast : X \Rightarrow Y\) and \(\Gamma^\ast : Y \Rightarrow X\) two correspondences. We say that \((\Gamma_\ast, \Gamma^\ast)\) is a set-valued Galois connection between \((X, \succ_X)\) and \((Y, \succ_Y)\) if

\[
 y \succ_Y a \text{ for some } a \in \Gamma_\ast(x) \quad \text{iff} \quad b \succ_X x \text{ for some } b \in \Gamma^\ast(y)
\]

for every \(x \in X\) and \(y \in Y\).

\(\text{a.}\) Let \(\mathcal{F}\) be a nonempty set of lower-adjoints between \((X, \succ_X)\) and \((Y, \succ_Y)\). For each \(f \in \mathcal{F}\), define \(\Phi_\mathcal{F} : X \Rightarrow Y\) and \(\Psi_\mathcal{F} : Y \Rightarrow X\) by

\[
 \Phi_\mathcal{F}(x) := \{f(x) : f \in \mathcal{F}\} \quad \text{and} \quad \Psi_\mathcal{F}(y) := \bigcup \{\max(f^{-1}(y^\dagger), \succ_X) : f \in \mathcal{F}\}.
\]

Show that \((\Phi_\mathcal{F}, \Psi_\mathcal{F})\) is a set-valued Galois connection between \((X, \succ_X)\) and \((Y, \succ_Y)\).

\(\text{b.}\) Let \(f : X \rightarrow Y\) be an order-preserving map such that \(f^{-1}\) is a weakly order-preserving correspondence. Show that \((f, f^{-1})\) is a set-valued Galois connection between \((X, \succ_X)\) and \((Y, \succ_Y)\).

\(\text{c.}\) Prove: \((\Gamma_\ast, \Gamma^\ast)\) is a set-valued Galois connection between \((X, \succ_X)\) and \((Y, \succ_Y)\) if \(\Gamma_\ast\) is lower order-preserving, \(\Gamma^\ast\) is upper order-preserving, and

\[
 a \in \Gamma_\ast(x) \quad \text{implies} \quad b \succ_X x \text{ for some } b \in \Gamma^\ast(a)
\]

and

\[
 b \in \Gamma^\ast(y) \quad \text{implies} \quad y \succ_Y a \text{ for some } a \in \Gamma_\ast(b)
\]

for any \((x, y) \in X \times Y\).

\(\text{d.}\) Given a poset \((Z, \succ)\), and subsets \(S\) and \(T\) of \(Z\) with \(S \subseteq T\), we say that \(S\) is \(\succ\)-cofinal in \(T\) if \(S \succ^\ast T\). Prove: If \((\Gamma_\ast, \Gamma^\ast)\) is a set-valued Galois connection between \((X, \succ_X)\) and \((Y, \succ_Y)\), then \(\Gamma^\ast(y)\) is \(\succ_X\)-cofinal in \(\{x \in X : y \succ_Y a \text{ for some } a \in \Gamma_\ast(x)\}\) for any \(y \in Y\).

\(\text{e.}\) Let \((\Gamma_\ast, \Gamma^\ast)\) be a set-valued Galois connection between \((X, \succ_X)\) and \((Y, \succ_Y)\). Where \(\bigvee\) stands for the \(\succ\)-supremum operator on \(2^X\), prove that

\[
 \Gamma_\ast(\bigvee S) = \bigvee \{\Gamma_\ast(s) : s \in S\}
\]

for every subset \(S\) of \(X\) such that \(\bigvee S\) exists.
4 An Application to Optimization Theory

In Section 9.2 of Chapter 1 we have seen that the maximizing points of a supermodular real map on a lattice form a sublattice of that lattice. We can actually say quite a bit more about this matter. Loosely speaking, if such a function depends on a parameter, under reasonably general conditions, its maximum solution sublattice responds “monotonically” to the alterations of that parameter. In this section we shall use the notion of strongly order-preserving correspondences to explore this issue.

Let $X$ and $\Theta$ be two nonempty sets, $f$ a real function on $X \times \Theta$ and $\Gamma : \Theta \rightrightarrows X$ a correspondence. The canonical parametric maximization problem is of the following form:

$$\text{Maximize } f(x, \theta) \text{ such that } x \in \Gamma(\theta).$$

Here, in the jargon of optimization theory, $X$ is the choice space, $\Theta$ is the parameter space, $f$ is the objective function, and $\Gamma$ is the constraint correspondence. As we allow both the objective function and the constraint correspondence to depend on the parameter $\theta$ here, the solutions of such a problem is bound to depend on $\theta$ as well. Indeed, provided that a solution for the problem exists for each $\theta$ in $\Theta$, we define the solution correspondence of our problem as the correspondence $\sigma : \Theta \rightrightarrows X$ with

$$\sigma(\theta) := \arg\max \{ f(x, \theta) : x \in \Gamma(\theta) \}, \quad (6)$$

that is,

$$\sigma(\theta) = \{ x \in \Gamma(\theta) : f(x, \theta) \geq f(y, \theta) \text{ for every } y \in \Gamma(\theta) \}.$$

The main problem of parametric optimization theory is to deduce the properties of $\sigma$ from those of $f$ and $\Gamma$. The following result, which was proved by Donald Topkis in 1978, illustrates what order theory may be able to do in this regard.

**Theorem 4.1.** Let $X$ be a topological space and $\succeq_X$ a partial order on $X$ such that $(X, \succeq_X)$ is a lattice. Let $(\Theta, \succeq)$ be a poset, and take any $f : X \times \Theta \to \mathbb{R}$ and $\Gamma : \Theta \rightrightarrows X$ such that

- $f(\cdot, \theta)$ is upper semicontinuous and $\succeq_X$-supermodular for each $\theta \in \Theta$;
- $f$ has increasing differences (relative to $\succeq_X$ and $\succeq$);
- $\Gamma$ is compact-valued and strongly order-preserving;
- $(\Gamma(\theta), \succeq_X)$ is a sublattice of $(X, \succeq_X)$ for each $\theta \in \Theta$. 

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Then, the correspondence $\sigma : \Theta \Rightarrow X$, defined by (6), is compact-valued and strongly order-preserving, and $(\sigma(\theta), \succ_X)$ is a sublattice of $(X, \succeq_X)$ for each $\theta \in \Theta$.

**Proof.** By the Baire Maximum Value Theorem (Section 1.8 of Appendix), $\sigma(\theta)$ is a nonempty compact subset of $\Gamma(\theta)$, and hence of $X$, for each $\theta$ in $\Theta$. Now, let $\alpha$ and $\beta$ be two elements of $\Theta$ such that $\alpha \succ \beta$. We wish to show that $\sigma(\alpha) \succ_X \sigma(\beta)$. To this end, take any $x$ in $\sigma(\alpha)$ and $y$ in $\sigma(\beta)$. Then, $x \vee y \in \Gamma(\alpha)$ and $x \wedge y \in \Gamma(\beta)$ because $\Gamma$ is strongly order-preserving. Therefore, $f(x, \alpha) \geq f(x \vee y, \alpha)$ and $f(y, \beta) \geq f(x \wedge y, \beta)$. Consequently, as $f(\cdot, \alpha)$ is $\succ_X$-supermodular and $f$ has increasing differences,

$$
0 \leq f(x, \alpha) - f(x \vee y, \alpha) \\
\leq f(x \wedge y, \alpha) - f(y, \alpha) \\
\leq f(x \wedge y, \beta) - f(y, \beta) \\
\leq 0
$$

so $f(x, \alpha) = f(x \vee y, \alpha)$ and $f(y, \beta) = f(x \wedge y, \beta)$, which means $x \vee y \in \sigma(\alpha)$ and $x \wedge y \in \sigma(\beta)$, as we sought. Finally, our third claim follows from Proposition 9.2.3 of Chapter 1.

In some special situations, one can even find an order-preserving function $\xi$ from $\Theta$ into $X$ such that $\xi(\theta)$ is a solution to the maximization problem at hand for each $\theta \in \Theta$.

**Corollary 4.2.** Set $(X, \succ_X) = (\mathbb{R}^n, \geq)$ in Theorem 4.1 and let $(\Theta, \succ)$, $f$ and $\Gamma$ be as in that result. Then, the correspondence $\sigma : \Theta \Rightarrow X$, defined by (6), admits an order-preserving selection.

**Proof.** By Theorem 4.1, $\sigma(\theta)$ is a nonempty compact subset of $\mathbb{R}^n$ such that $(\sigma(\theta), \geq)$ is a sublattice of $(\mathbb{R}^n, \geq)$, for any $\theta \in \Theta$. By what we have shown in Example 8.2.5 of Chapter 1, therefore, $(\sigma(\theta), \geq)$ is a subcomplete sublattice of $(\mathbb{R}^n, \geq)$, for each $\theta \in \Theta$. As $\sigma$ is strongly order-preserving (Theorem 4.1), therefore, applying Corollary 3.2.2 completes the proof.

**Exercises**

4.1. (Topkis) Let $(X, \succ_X)$, $(\Theta, \succ)$, $f$ and $\Gamma$ be as in Theorem 4.1, and assume, in addition, that $f$ has strictly increasing differences (relative to $\succ_X$ and $\succ$). Prove that

$$
\alpha \succ \beta \quad \text{implies} \quad x \succ_X y \quad \text{for every} \quad (x,y) \in \Gamma(\alpha) \times \Gamma(\beta),
$$

where $\sigma : \Theta \Rightarrow X$ is as defined by (6).
4.2. (Topkis) Let \((X, \succ_X)\) and \((\Theta, \succ)\) be two lattices, and \((Z, \supseteq)\) a sublattice of the product of these two lattices. Let \(f : Z \rightarrow \mathbb{R}\) be a \(\supseteq\)-supermodular function such that \(\sigma(\theta) := \{x \in X : (x, \theta) \in Z\text{ and } f(x, \theta) \geq f(y, \theta)\text{ for every } (y, \theta) \in Z\}\) is a nonempty set for each \(\theta \in \Theta\). Prove that the correspondence that map each \(\theta\) in \(\Theta\) to \(\sigma(\theta)\) is strongly order-preserving.

4.3. (Migrom-Shannon) Let \((X, \succ_X)\), \((\Theta, \succ)\), and \(\Gamma\) be as in Theorem 4.1. Let \(f : X \times \Theta \rightarrow \mathbb{R}\) be a function such that \(f(\cdot, \theta)\) is upper semicontinuous and \(\succ_X\)-quasi-supermodular for each \(\theta \in \Theta\). (Recall Exercise 9.9 of Chapter 1.) In addition, assume that, for every \((x, \alpha)\) and \((y, \beta)\) in \(X \times \Theta\) with \(x \succ_X y\) and \(\alpha \succ \beta\), we have \(f(x, \alpha) > f(y, \alpha)\) whenever \(f(x, \beta) \geq f(y, \beta)\). Prove: The correspondence \(\sigma : \Theta \rightarrow X\), defined by (6), is compact-valued and strongly order-preserving, and \((\sigma(\theta), \succ_X)\) is a sublattice of \((X, \succ_X)\) for each \(\theta \in \Theta\).