Chapter 1

Metric Spaces

1 Metric Spaces

Metric spaces are the friendliest sorts of topological spaces that still allow for a nontrivial analysis of the notions of convergence and continuity, which are at the heart of point set topology. Besides, unlike abstract topological spaces, they possess a flair of being somewhat concrete, and serve as workhorses for various branches of mathematical analysis and its applications.

If only as a prelude to the coming attractions, we will work exclusively with metric spaces in this and the following two chapters. As such, these chapters have more of a “real analysis” feel to them as opposed to topology. Indeed, full on point set topology will not kick in until we get to Chapter 4. But make no mistake about it. These chapters will cultivate a decent understanding of a good number of fundamental topological notions far before then.

1.1 First Impressions

Our story begins with the following axiomatic definition.

**Definition.** Let $X$ be a nonempty set. A function $d : X \times X \to \mathbb{R}_+$ that satisfies the following properties is called a **distance function** (or a **metric**) on $X$: For any $x, y, z \in X$,

- **(Separation)** $d(x, y) = 0$ if and only if $x = y$;
- **(Symmetry)** $d(x, y) = d(y, x)$; and
- **(Triangle Inequality)** $d(x, y) \leq d(x, z) + d(z, y)$. 

If $d$ is a distance function on $X$, we say that $(X, d)$ is a **metric space**.

**Warning.** Let $d$ be an $\mathbb{R}_+\text{-valued}$ map on $X \times X$ such that $d(x, x) = 0$ for every $x \in X$. If this map satisfies the symmetry and triangle inequality properties of being a metric, we refer to it as a **semimetric** on $X$. (Thus, what separates a semimetric on $X$ from being a metric is that it may assess the distance between two distinct points in $X$ as zero.) Semimetrics are commonly encountered in real analysis, and we too will have occasion to work with them later in the text.

We are primed to think of the distance between two points $x$ and $y$ on the real line as $|x - y|$. Thus the map $(x, y) \mapsto |x - y|$ tells us how far apart any two elements of $\mathbb{R}$ are. This function is nonnegative-valued and surely satisfies the three axioms of being a metric. By way of abstraction, the notion of distance function is built *only* on these properties. And you may rightly ask, “Why these properties?” There is no easy answer to this question; the proof is really in the pudding in this case. We will see below that, quite surprisingly, the combination of these simple properties is strong enough to provide an environment for building satisfactory theories of continuous functions, functional equations, and optimization, among others.

**Notation.** When the metric under consideration is apparent from the context, it is customary to dispense with the notation $(X, d)$, and refer to $X$ as a metric space. We also adhere to this convention here (and spare the notation $d$ for a generic metric on $X$). But when we feel that there is a danger of confusion, or we endow $X$ with a particular metric $d$, we shall revert back to the more descriptive notation $(X, d)$.

Strictly speaking, we cannot talk of “an element of a metric space” because such a space is not a set, but a set and a certain type of function put together. But this is splitting hairs. In what follows, depending on the context, we will at times talk about a metric space as if this space is a set. So, formally speaking, by “an element of $(X, d)$,” or by “an element of the metric space $X,” we simply mean an element of the set $X$. In fact, in concert with the geometric intuition of things, we will often refer to such an element as a **point** in the space $(X, d)$ (or in $X$).

**Historical Note.** In the 1897 International Congress of Mathematicians, the great Jacques Hadamard gave a plenary address in which he suggested that studying abstract spaces whose points are functions may not be without merit. (In particular, Hadamard (rightly, and way ahead of his time) thought this sort of study would be of great use for the theory of partial differential equations of mathematical physics.) Hadamard’s proposal was soon followed by the introduction of the axiomatic notion of “metric space” by his student Maurice Fréchet in his seminal 1906 doctoral dissertation. This line of abstraction turned out to be extremely fruitful, and eventually led to the development of the fields of functional analysis and general topology.

Originally, Fréchet referred to metric spaces as “(E) spaces,” where (E) was motivated by the French word “écart” which means distance. The term “metric space”
was coined first by Felix Hausdorff. Indeed, most of the basic terms we use today in the context of metric spaces are due to either Fréchet or Hausdorff.

Before we move on to examples, let us make note of a very useful property of distance functions. Let \( X \) be a metric space (with metric \( d \)). Then, for any \( x, y, z \in X \),

\[
|d(x, y) - d(y, z)| \leq d(x, z).
\]

This is often called the reverse triangle inequality, and it is an easy consequence of the symmetry and triangle inequality properties of \( d \). Indeed, for any \( x, y, z \in X \), these properties entail \( d(x, y) - d(y, z) \leq d(x, z) \), and replacing the roles of \( x \) and \( z \) in this inequality (and using symmetry again), we get \( d(y, z) - d(x, y) \leq d(x, z) \), and hence the reverse triangle inequality. We will use this inequality numerous times in what follows.

### 1.2 Examples

Let us look at some standard examples of metric spaces.

**Example 1.1.** Let \( X \) be any nonempty set. We can make \( X \) a metric space by using the metric \( d : X \times X \to \mathbb{R}_+ \) which is defined by

\[
d(x, y) := \begin{cases} 
1, & \text{if } x \neq y \\
0, & \text{if } x = y.
\end{cases}
\]

It is easy to check that \((X, d)\) is indeed a metric space. Here \( d \) is called the discrete metric on \( X \); and \((X, d)\) is called a discrete space. In this space, the distance between any two distinct points is the same.

**“Euclidean” Examples**

**Example 1.2.** (\( \mathbb{R} \) as a metric space) It is readily verified that the map \((x, y) \mapsto |x - y|\) is indeed a distance function on \( \mathbb{R} \) — this function is called the absolute value metric. When we talk about \( \mathbb{R} \) as a metric space, we have this metric in mind, unless otherwise is explicitly stated.

**Example 1.3.** (\( \mathbb{R} \) as a metric space) We may turn the set \( \mathbb{R} \) of all extended real numbers into a metric space in a way that is duly consistent with the metrization of \( \mathbb{R} \). Consider the map \( f : \mathbb{R} \to [-1, 1] \), defined as

\[
f(-\infty) := -1, \quad f(\infty) := 1, \quad \text{and} \quad f(t) := \frac{x}{1 + |x|}
\]

for every \( x \in \mathbb{R} \), and then define \( d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}_+ \) by \( d(x, y) := |f(x) - f(y)| \). It is easily checked that \( d \) is a metric on \( \mathbb{R} \). There is a sense in which this metrization makes \( \mathbb{R} \) “identical” to \([-1, 1]\), and in addition, \( \mathbb{R} \) lives within \( \mathbb{R} \) in a fairly natural manner. These points are a bit vague at the moment; we will make them precise in the next chapter.
Example 1.4. Given any positive integer \( n \), there are various ways of metrizing \( \mathbb{R}^n \). Indeed, \( (\mathbb{R}^n, d_p) \) is a metric space for each \( p \in [1, \infty] \), where \( d_p : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+ \) is defined by

\[
d_p(x, y) := \left( \sum_{i=1}^{n} |x_i - y_i|^p \right)^{1/p} \quad \text{for} \ p \in [1, \infty),
\]

and

\[
d_p(x, y) := \max_{i=1,\ldots,n} |x_i - y_i| \quad \text{for} \ p = \infty.
\]

It is easy to check that \( d_1 \) is a metric on \( \mathbb{R}^n \). We can also easily verify that \( d_\infty \) is a metric on \( \mathbb{R}^n \), but we will prove something a bit more general than this in a bit, so we omit the needed argument here. On the other hand, while \( d_p \) obviously satisfies the separation and symmetry axioms of being a distance function when \( p \in (1, \infty) \), the verification of the triangle inequality in this case is not a trivial matter. It rather follows from the following celebrated result of Hermann Minkowski:

**Minkowski’s Inequality 1.** Let \( n \) be a positive integer, and take any real numbers \( a_1, \ldots, a_n, b_1, \ldots, b_n \). Then, for any \( p \in [1, \infty) \),

\[
\left( \sum_{i=1}^{n} |a_i + b_i|^p \right)^{1/p} \leq \left( \sum_{i=1}^{n} |a_i|^p \right)^{1/p} + \left( \sum_{i=1}^{n} |b_i|^p \right)^{1/p}.
\]

(We will prove this result shortly.) Indeed, for any \( n \)-vectors \( x, y \) and \( z \), and any \( p \in [1, \infty) \), setting \( a_i := x_i - y_i \) and \( b_i := y_i - z_i \) for each \( i = 1, \ldots, n \), in Minkowski’s Inequality 1 yields the triangle inequality for \( d_p \).

The space \( (\mathbb{R}^n, d_2) \) is called the \textbf{\( n \)-dimensional Euclidean space} in analysis, and \( d_2 \) is often referred to as the \textbf{Euclidean metric}. When we refer to \( \mathbb{R}^n \) in the sequel without specifying a particular distance function, you should understand that we view this set as metrized by \( d_2 \). That is to say, the notation \( \mathbb{R}^n \) is spared for the \( n \)-dimensional Euclidean space. If we wish to endow \( \mathbb{R}^n \) with a metric different than \( d_2 \), we will be explicit about it.

![Figure 1.1](attachment:image.png)
Notation. We denote the metric space \((\mathbb{R}^n, d_p)\) as \(\mathbb{R}^{n,p}\) for any \(p \in [1, \infty]\). However, almost always, we use the notation \(\mathbb{R}^n\) instead of \(\mathbb{R}^{n,2}\).

Each of the metrics \(d_p\) endows \(\mathbb{R}^n\) with a different type of metric geometry. For instance, the shape of the unit “circle” \(\{x \in \mathbb{R}^2 : d_p(0, x) = 1\}\) in \(\mathbb{R}^2\) depends very much on what \(p\) we use here. To illustrate, Figure 1.1 depicts this “circle” for the choices 1, 2 and \(\infty\). (We will later see, however, that for purely topological purposes, these metrics are not really different from each other.)

Classical Sequence Spaces

Example 1.5. For any \(p \in [1, \infty)\), we define

\[
\ell_p := \left\{ x \in \mathbb{R}^\infty : \sum_{i=1}^\infty |x_i|^p < \infty \right\}.
\]

(Here \(x\) is viewed as the sequence \((x_1, x_2, ...), a\) convention that we will adopt in the entire text.) We metrize \(\ell_p\) by means of the metric \(d_p : \ell_p \times \ell_p \to \mathbb{R}_+\) with

\[
d_p(x, y) := \left( \sum_{i=1}^\infty |x_i - y_i|^p \right)^{1/p}.
\]

(When we speak of \(\ell_p\) as a metric space, we always have this metric in mind!) The separation and symmetry properties of being a metric are obviously satisfied by any \(d_p\), but it is not obvious if \(d_p\) is well-defined as a real-valued function, and if it satisfies the triangle inequality. These facts follow from the following generalization of Minkowski’s Inequality 1:

Minkowski’s Inequality 2. For any real sequences \((a_m)\) and \((b_m)\), and any \(p \in [1, \infty)\),

\[
\left( \sum_{i=1}^\infty |a_i + b_i|^p \right)^{1/p} \leq \left( \sum_{i=1}^\infty |a_i|^p \right)^{1/p} + \left( \sum_{i=1}^\infty |b_i|^p \right)^{1/p}.
\]

We will prove this inequality at the end of this subsection.

Example 1.6. By \(\ell_\infty\), we mean the set of all bounded real sequences, that is,

\[
\ell_\infty := \left\{ x \in \mathbb{R}^\infty : \sup_{m \in \mathbb{N}} |x_m| < \infty \right\}.
\]

It is implicitly understood that this set is endowed with the metric \(d_\infty : \ell_\infty \times \ell_\infty \to \mathbb{R}_+\) with

\[
d_\infty (x, y) := \sup_{m \in \mathbb{N}} |x_m - y_m|.
\]

That \(d_\infty\) is indeed a metric will be verified below. This metric is called the \textbf{sup-metric} on the set of all bounded real sequences.
Before we leave this example let us stress that any \( \ell_p \) space is smaller than the set \( \mathbb{R}^\infty \) of all real sequences, because the members of such a space are real sequences that are either bounded or that satisfy some form of a summability condition (which ensures that \( d_p \) is real-valued). Indeed, no \( d_p \) defines a distance function on the entire \( \mathbb{R}^\infty \). But this does not mean that we cannot metrize the set of all real sequences in a useful way. We can, and we will, later in this chapter.

A First Example of a Function Space

**Example 1.7.** Let \( X \) be a nonempty set. By \( B(X) \), we mean the set of all bounded real functions defined on \( X \), that is,

\[
B(X) := \left\{ f \in \mathbb{R}^X : \sup_{x \in X} |f(x)| < \infty \right\}.
\]

We will always think of this space as metrized by the **sup-metric** \( d_\infty : B(X) \times B(X) \to \mathbb{R}_+ \) which is defined by

\[
d_\infty (f, g) := \sup_{x \in X} |f(x) - g(x)|.
\]

It is easy to see that \( d_\infty \) is real-valued. Indeed, for any \( f, g \in B(X) \),

\[
d_\infty (f, g) \leq \sup_{x \in X} |f(x)| + \sup_{x \in X} |g(x)| < \infty.
\]

It is also readily checked that \( d_\infty \) satisfies the first two requirements of being a distance function. As for the triangle inequality, all we need is to invoke the corresponding property of the absolute value function. After all, if \( f, g, h \in B(X) \), then

\[
|f(x) - g(x)| \leq |f(x) - h(x)| + |h(x) - g(x)|
\]

\[
\leq \sup_{y \in X} |f(y) - h(y)| + \sup_{y \in X} |h(y) - g(y)|
\]

\[
= d_\infty (f, h) + d_\infty (h, g)
\]

for any \( x \in X \), so

\[
d_\infty (f, g) = \sup_{x \in X} |f(x) - g(x)| \leq d_\infty (f, h) + d_\infty (h, g).
\]

Given that a sequence and/or an \( n \)-vector can always be thought of as special functions, it is plain that \( B\{1, \ldots, n\} \) coincides with \( \mathbb{R}^{n, \infty} \) (for any \( n \in \mathbb{N} \)) while \( B(\mathbb{N}) \) coincides with \( \ell_\infty \). Therefore, the inequality we just established proves in one stroke that both \( \mathbb{R}^{n, \infty} \) and \( \ell_\infty \) are metric spaces.

**Remark.** Obviously, distance functions need not be bounded. However, given any metric space \((X, d)\), we can always transform \( d \) into a bounded metric in
a way that leaves many of the properties of \((X, d)\) intact. A particularly easy way of doing this is to define \(D : X \times X \to \mathbb{R}_+\) as
\[
D := \min\{1, d\}.
\]
It is readily checked that \(D\) is a \([0, 1]\)-valued metric on \(X\). As we proceed further, it will become clear that there is a good sense in which \((X, d)\) and \((X, D)\) can be thought of as “equivalent” in terms of certain characteristics (and not so in terms of others).

**Proofs of Minkowski’s Inequalities**

The final order of business in this subsection is to prove Minkowski’s Inequalities which we have invoked above to verify that \(\mathbb{R}^{n,p}\) and \(\ell_p\) are metric spaces for any \(p \in [1, \infty)\). A moment’s reflection shows that Minkowski’s Inequality 1 is a special case of Minkowski’s Inequality 2, so all we need is to establish the latter inequality.

**Proof of Minkowski’s Inequality 2.** Take any real sequences \((a_m)\) and \((b_m)\), and fix any \(p \in [1, \infty)\). If either \(\sum_{i=1}^{\infty} |a_i|^p\) or \(\sum_{i=1}^{\infty} |b_i|^p\) is \(\infty\), then (1) becomes trivial, so we assume that these numbers are finite. (1) is also trivially true if either \((a_m)\) or \((b_m)\) equals \((0, 0, \ldots)\), so we focus on the case where
\[
\alpha := \left(\sum_{i=1}^{\infty} |a_i|^p\right)^{1/p} \quad \text{and} \quad \beta := \left(\sum_{i=1}^{\infty} |b_i|^p\right)^{1/p}
\]
are positive real numbers.

For every positive integer \(m\), let us put \(c_m := \frac{1}{\alpha} |a_m|\) and \(d_m := \frac{1}{\beta} |b_m|\). (Notice that \(\sum_{i=1}^{\infty} c_i^p = 1 = \sum_{i=1}^{\infty} d_i^p\).) Using the triangle inequality for the absolute value function, and the fact that \(t \mapsto t^p\) is an increasing map on \(\mathbb{R}_+\), we find
\[
|a_i + b_i|^p \leq (|a_i| + |b_i|)^p = (\alpha c_i + \beta d_i)^p = (\alpha + \beta)^p \left(\frac{\alpha}{\alpha + \beta} c_i + \frac{\beta}{\alpha + \beta} d_i\right)^p
\]
for each \(i\). To simplify the notation, we put \(\lambda := \frac{\alpha}{\alpha + \beta}\), and note that \(\frac{\beta}{\alpha + \beta} = 1 - \lambda\). But \(t \mapsto t^p\) is a convex map on \(\mathbb{R}_+\); see Section 2.6 of Appendix. Consequently,
\[
(\lambda c_i + (1 - \lambda)d_i)^p \leq \lambda c_i^p + (1 - \lambda)d_i^p,
\]
for each \(i\), and hence
\[
|a_i + b_i|^p \leq (\alpha + \beta)^p \left(\lambda c_i^p + (1 - \lambda)d_i^p\right)
\]
for each \(i\). Summing over \(i\), then,
\[
\sum_{i=1}^{\infty} |a_i + b_i|^p \leq (\alpha + \beta)^p \left(\lambda \sum_{i=1}^{\infty} c_i^p + (1 - \lambda) \sum_{i=1}^{\infty} d_i^p\right)
\]
Thus \(\sum_{i=1}^{\infty} |a_i + b_i|^p \leq (\alpha + \beta)^p\) which is equivalent to (1).

**Remark.** Take any real number \(p > 1\). Then, (1) holds as an equality iff either \((a_m) = (0, 0, \ldots)\) or \((b_m) = \lambda (a_m)\) for some \(\lambda \geq 0\). The proof is left as an exercise.
1.3 Normed Linear Spaces

Let $X$ be a (real) linear space. An $\mathbb{R}^+$-valued function $\| \cdot \|$ on $X$ is said to be a norm on $X$ if for every $x, y \in X$ and $\lambda \in \mathbb{R}$, we have (i) $\|x\| = 0$ iff $x$ is the origin $0$ of $X$; (ii) (absolute homogeneity) $\|\lambda x\| = |\lambda| \|x\|$; and (iii) (subadditivity) $\|x + y\| \leq \|x\| + \|y\|$. The basic viewpoint of vector calculus is to regard a “vector” $x$ in a linear space as a directed line segment that begins at 0 and ends at $x$. This allows one to think of the “length” of a vector in a natural way. For instance, we think of the length of a vector in $\mathbb{R}^n$ as the distance between this vector and the origin, and as you would expect, $x \mapsto d_2(x,0)$ defines a norm on $\mathbb{R}^n$. Just as the notion of metric generalizes the geometric notion of “distance,” therefore, the notion of norm generalizes that of “length” of a vector.

An ordered pair $(X, \| \cdot \|)$ is called a normed linear space if $X$ is a linear space and $\| \cdot \|$ is a norm on $X$. When the norm under consideration is transparent (or is arbitrary), we denote such a space simply as $X$; and talk about $X$ as if it is a set, or a linear space, or a normed linear space, depending on the context.

Historical Note. Normed linear spaces were used, often implicitly, before an axiomatic definition for them was formally introduced. The axiomatic definition was given first by Frédéric Riesz in 1918 who at the time was after a general theory of integral equations. However, Riesz did not do much in the way of developing the theory of such spaces. In 1921 Eduard Helly took a stab at doing precisely this, but Helly’s work was in the context of some particular sequence spaces. All in all, making a solid attribution in this regard is quite difficult. What is clear, however, is that the general theory of normed linear spaces was thoroughly investigated first by the great Stefan Banach in 1922 (but Banach studied such spaces under an additional assumption that we will look at only in Chapter 3). It was mainly Banach’s work that brought such spaces to the center of mathematical analysis. We will talk more about Banach later in the text.

Examples of normed linear spaces abound. In fact, most of the metric spaces we have considered in the previous section are normed linear spaces.

Example 1.9. For any $p \in [1, \infty]$, the map $\| \cdot \|_p$ on $\mathbb{R}^n$, defined by $\|x\|_p := d_p(x,0)$, is a norm on $\mathbb{R}^n$. (Here 0 is the $n$-vector of 0s.) For $p \in [1, \infty)$, we refer to $\| \cdot \|_p$ as the $p$-norm on $\mathbb{R}^n$, but the 2-norm is often called the Euclidean norm. On the other hand, $\| \cdot \|_\infty$ is called the sup-norm on $\mathbb{R}^n$. (Unless otherwise is explicitly mentioned, we always consider $\mathbb{R}^n$ as a normed linear space relative to the Euclidean norm.)

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1. We assume in this text that the reader is familiar with linear (vector) spaces. Incidentally, all linear spaces we will consider will be over the field of real numbers.


3. In each of the following examples, we understand that the addition and scalar multiplication operations are defined coordinatewise. For instance, by the sum of two sequences $(x_m)$ and $(y_m)$, we understand the sequence $(x_m + y_m)$. Similarly, the product of a real number $\lambda$ and a real function $f$ (on any given nonempty set) is understood to be the function $\lambda f$ which maps any one point $x$ on the domain of $f$ to the real number $\lambda f(x)$. 

8
Example 1.10. Let us denote the zero sequence \((0, 0, \ldots)\) by \(0\), and fix an arbitrary \(p \in [1, \infty)\). For any \(\lambda \in \mathbb{R}\), and any two points \(x\) and \(y\) in \(\ell_p\), the triangle inequality for the absolute value function implies

\[
\sum_{i=1}^{m} |\lambda x_i + y_i|^p \leq |\lambda| \sum_{i=1}^{m} |x_i|^p + \sum_{i=1}^{m} |y_i|^p \leq |\lambda| \sum_{i=1}^{\infty} |x_i|^p + \sum_{i=1}^{\infty} |y_i|^p < \infty,
\]

and thus letting \(m \uparrow \infty\) shows that \(\lambda x + y \in \ell_p\). Thus, \(\ell_p\) is a linear subspace of the linear space \(\mathbb{R}^\infty\) of real sequences (under the coordinatewise defined operations of addition and scalar multiplication). Moreover, the map of the linear space

\[\|\cdot\|_p \colon \ell_p \to \mathbb{R}_+\]

and thus letting \(m \uparrow \infty\) shows that \(\lambda x + y \in \ell_p\). Thus, \(\ell_p\) is a linear subspace of the linear space \(\mathbb{R}^\infty\) of real sequences (under the coordinatewise defined operations of addition and scalar multiplication). Moreover, the map \(\|\cdot\|_p \colon \ell_p \to \mathbb{R}_+\), defined by \(\|x\|_p := d_p(x, 0)\), is a norm on \(\ell_p\)—it is called the \(p\)-norm on \(\ell_p\). (Indeed, while \(\|\cdot\|_p\) obviously satisfies properties (i) and (ii) of being a norm, Minkowski’s Inequality 2 says precisely that \(\|\cdot\|_p\) is subadditive on \(\ell_p\).) It is commonplace to consider \(\ell_p\) as a normed linear space relative to this norm.

Example 1.11. For any nonempty set \(X\), \(B(X)\) is a linear space. (Why?) This linear space is normed by \(\|\cdot\|_\infty : B(X) \to \mathbb{R}_+, \text{ where } \|f\|_\infty := d_\infty(f, 0), \text{ with } 0 \text{ being the zero function on } X\), that is,

\[\|f\|_\infty := \sup_{x \in X} |f(x)|.\]

For obvious reasons, \(\|\cdot\|_\infty\) is called the \textbf{sup-norm}. (Note. As usual, we use the notation \(\ell_\infty\) instead of \(B(\mathbb{N})\) in what follows.)

Normed linear spaces provide a natural playground for functional analysis instead of topology. However, we should note that every normed linear space gives rise to a metric space in a natural manner. Indeed, given a normed linear space \((X, \|\cdot\|)\), the real map \(d_{\|\cdot\|} \colon \ell_p \times \ell_p \to \mathbb{R}_+\) defined on \(X \times X\) by

\[
d_{\|\cdot\|}(x, y) := \|x - y\|\tag{2}
\]

is a metric on \(X\). We always think of \((X, \|\cdot\|)\) as a metric space with respect to this distance function.

\textbf{Notation.} In the context of \(\ell_p\), where \(p \in [1, \infty]\), it is far more common to denote the distance between two real sequences \(x\) and \(y\) in \(\ell_p\) as \(\|x - y\|_p\) instead of \(d_p(x, y)\). (Here, of course, \(x - y\) is the sequence \((x_1 - y_1, x_2 - y_2, \ldots)\).) Similarly, in the context of \(B(X)\), we write \(\|f - g\|_\infty\) instead of \(d_\infty(f, g)\) for any two bounded real maps \(f\) and \(g\) on \(X\). In particular, where \(\mathbf{0}\) is the real map on \(X\) that equals 0 everywhere, we write \(\|f\|_\infty\) instead of \(d_\infty(f, \mathbf{0})\).

\textbf{Warning.} One cannot use (2) to derive a norm from a metric in general. That is, \(x \mapsto d(x, \mathbf{0})\) need not be a norm on a linear space \(X\) even if \(d\) is a metric on \(X\). For instance, if \(d\) is the discrete metric on \(\mathbb{R}\), then \(d(\cdot, \mathbf{0})\) is not a norm on \(\mathbb{R}\). Thus, insofar as metric analysis is concerned, metric spaces are more general than normed linear spaces.
Exercises

1.1. (The Counting Metric) Let $X$ be a finite set and $X$ a nonempty collection of subsets of $X$. Define $d : X \times X \to \mathbb{R}_+$ by $d(A, B) := |A \Delta B|$. Show that $d$ is a metric on $X$.

1.2. Let $X$ be a nonempty set and $f : X \to \mathbb{R}_+$ a function such that $f(x) = 0$ for at most one $x$ in $X$. Define $d : X \times X \to \mathbb{R}_+$ by $d(x, y) := f(x) + f(y)$ for every distinct $x, y \in X$, and by $d(x, x) := 0$ for every $x \in X$. Show that $d$ is a metric on $X$.

1.3. Let $X$ be a nonempty set and $f : X \times X \to \mathbb{R}_+$ a function such that for every $x, y, z \in X$, (i) $f(x, y) = 0$ iff $x = y$, and (ii) $f(x, y) \leq f(x, z) + f(z, y)$. Define $d : X \times X \to \mathbb{R}_+$ by $d(x, y) := f(x, y) + f(y, x)$, and show that $d$ is a metric on $X$.

1.4. Let $X$ and $Y$ be two nonempty sets and $u : X \times Y \to \mathbb{R}$ a bounded function. Consider the map $d : X \times X \to \mathbb{R}_+$ defined by

\[d(x_1, x_2) := \sup_{y \in Y} |u(x_1, y) - u(x_2, y)|.\]

Show that $d$ is a metric on $X$. (This metric is called the Helly metric.)

1.5. Define $d : \mathbb{N} \times \mathbb{N} \to \mathbb{R}_+$ by $d(i, j) := 1 + \max\{2^{-i}, 2^{-j}\}$ if $i$ and $j$ are distinct, and $d(i, i) := 0$. Show that $(\mathbb{N}, d)$ is a metric space.

1.6. Let $X$ be a nonempty set. Suppose that there exists a function $d : X \times X \to \mathbb{R}_+$ which satisfies the separation and symmetry properties of being a metric, and in addition, has the property that $d(x, y) \geq d(x, z) + d(z, y)$ for every $x, y, z \in X$. Show that $X$ must then be a singleton.

1.7. (Ultrametric Spaces) Let $X$ be a nonempty set, and $d : X \times X \to \mathbb{R}_+$ a function which satisfies the separation and symmetry properties of being a metric, and in addition, has the property that $d(x, y) \leq \max\{d(x, z), d(z, y)\}$ for every $x, y, z \in X$. Such a function is said to be an ultrametric on $X$, and when $d$ is an ultrametric on $X$, we refer to $(X, d)$ as an ultrametric space. Clearly, every ultrametric space is a metric space. Give an example to show that the converse of this is false.

1.8. Let $X$ be a nonempty set. For any distinct $x$ and $y$ in $X^\infty$, let $k(x, y)$ be the first term at which the sequences $x$ and $y$ differ. Consider the function $d : X^\infty \times X^\infty \to \mathbb{R}_+$ defined by $d(x, y) := 1/k(x, y)$ for every distinct $x, y \in X^\infty$, and by $d(x, x) := 0$ for every $x \in X^\infty$. Show that $d$ is an ultrametric on $X^\infty$.

1.9. If $(X, d)$ and $(X, D)$ are metric spaces, is $(X, \max\{d, D\})$ necessarily a metric space? How about $(X, \min\{d, D\})$?

1.10. Let $(X, d)$ be a metric space and $f : \mathbb{R}_+ \to \mathbb{R}$ a subadditive and strictly increasing function with $f(0) = 0$. (Here “subadditive” means that $f(a + b) \leq f(a) + f(b)$ for every $a, b \in \mathbb{R}_+$.) Show that $(X, f \circ d)$ is a metric space.

1.11. Let $(X, d)$ be a metric space and use the previous exercise to show that $\frac{d}{1 + t}$ is a metric on $X$. (Hint: Consider the real function $f$ on $\mathbb{R}_+$ defined by $f(t) := t/(1 + t)$. Next, fix an arbitrary $s$ in $\mathbb{R}_+$, define the real map $g$ on $\mathbb{R}_+$ by $g(t) := f(s + f(t) - f(s + t)$, and check that $g'(t) > 0$, so that $g(t) > g(0) = 0$, for any $t > 0$.)
1.4 Construction of Metrics by Chains

There are situations in which one is given a numerical measure of separation between any two points of a nonempty set $X$, that is, a map $f : X \times X \to \mathbb{R}_+$ that satisfy the separation and symmetry properties of being a metric (but not the triangle inequality). In such cases, there is a standard method of obtaining a semimetric out of $f$. Indeed, it is easy to check that $d : X \times X \to \mathbb{R}_+$, defined by

$$d(x, y) := \inf \sum_{i=1}^{k} f(x_{i-1}, x_i)$$

where the infimum is taken over all finite subsets $\{x_0, \ldots, x_k\}$ of $X$ with $x_0 = x$ and $x_k = y$, is a semimetric on $X$. (Note. If $f$ is a metric on $X$, then $d = f$.)

However, in general, the infimum above may be zero for distinct $x$ and $y$, so we need to impose further conditions on $f$ to end up with a metric via this method. Obviously, these conditions must be weaker than the triangle inequality, for otherwise $f$ is itself a metric and there is no need for an additional construction. The following variants of the triangle inequality will be useful for this purpose.

**Definition.** Let $X$ be a nonempty set. A function $f : X \times X \to \mathbb{R}_+$ is said to satisfy the **weak triangle inequality** if

$$\max\{f(x, y), f(y, z)\} \leq \varepsilon \quad \text{implies} \quad f(x, z) \leq 2\varepsilon$$

for every $x, y, z \in X$ and $\varepsilon \geq 0$. It is said to satisfy the **quadrilateral inequality** if

$$\max\{f(x_1, x_2), f(x_2, x_3), f(x_3, x_4)\} \leq \varepsilon \quad \text{implies} \quad f(x_1, x_4) \leq 2\varepsilon$$

(3)

for every $x_1, \ldots, x_4 \in X$ and $\varepsilon > 0$.

It is easily verified that quadrilateral inequality implies the weak triangle inequality for any map $f : X \times X \to \mathbb{R}_+$ with $f(x, x) = 0$ for every $x \in X$. (Proof. Let $f$ satisfy the quadrilateral inequality, and take any $x, y, z \in X$ and $\varepsilon \geq 0$ with $\max\{f(x, y), f(y, z)\} \leq \varepsilon$. If $\varepsilon > 0$, then we set $x_1 = x$, $x_2 = y$ and $x_3 = x_4 = z$ in (3) to find $f(x, z) \leq 2\varepsilon$. If $\varepsilon = 0$, then $f(x, y) = 0$ and $f(y, z) = 0$, so this argument shows that $f(x, z) \leq 2\delta$ for every $\delta > 0$, which is possible if $f(x, z) = 0$.) In turn, the weak triangle inequality for $f$ implies that

$$f(x_0, x_1) = \cdots = f(x_{k-1}, x_k) = 0 \quad \text{implies} \quad f(x_0, x_k) = 0$$

for any positive integer $k$ and $x_0, \ldots, x_k \in X$. (Proof. If $k = 2$, this is immediate from the weak triangle inequality (with $\varepsilon = 0$). The rest of the claim follows by induction on $k$.)

With this brief preparation under our belt, we now prove that the method we described above yields a metric from a numerical measure of separation $f$ between points in $X$, provided that $f$ satisfies the quadrilateral inequality.

**Theorem 1.1.** Let $X$ be a nonempty set, and $f : X \times X \to \mathbb{R}_+$ a map that satisfies the separation and symmetry properties of being a metric as well as the quadrilateral inequality for some $\varepsilon > 0$. Then $d : X \times X \to \mathbb{R}_+$, defined by

$$d(x, y) := \sum_{i=1}^{k} f(x_{i-1}, x_i)$$

where the sum is taken over all finite subsets $\{x_0, \ldots, x_k\}$ of $X$ with $x_0 = x$ and $x_k = y$, is a metric on $X$. (Proof. It is easily verified that $d$ satisfies the triangle inequality, the separation and symmetry properties of being a metric as well as the quadrilateral inequality for any $\varepsilon > 0$. Indeed, it is enough to prove that

$$d(x, y) \leq 2d(x, z)$$

for all $x, y, z \in X$. (Proof. If $x = z$, this is immediate from the weak triangle inequality, (3). If $x \neq z$, then $x \neq y$, and we set $x_1 = x$, $x_2 = y$ and $x_3 = x$ in (3) to find $d(x, y) \leq 2d(x, z)$.)

This section is not particularly advanced, but it will not be needed until Section 2 of Chapter 12.
inequality. Then, the map $d : X \times X \to \mathbb{R}_+$ defined by

$$d(x, y) := \inf \sum_{i=1}^{k} f(x_{i-1}, x_i)$$

where the infimum is taken over all finite subsets $\{x_0, \ldots, x_k\}$ of $X$ with $x_0 = x$ and $x_k = y$, is a metric on $X$ such that $\frac{1}{2}f \leq d \leq f$.

**Proof.** Throughout our proof, for any positive integer $k$ and $x, y \in X$, we let $P_k(x, y)$ stand for all $k$-element subsets $\{x_0, \ldots, x_k\}$ of $X$ with $x_0 = x$ and $x_k = y$, and for any $\{x_0, \ldots, x_k\} \in P_k(x, y)$, define

$$|x_0, \ldots, x_k| := \sum_{i=1}^{k} f(x_{i-1}, x_i).$$

Where $P(x, y) := \bigcup_{k=2}^{\infty} P_k(x, y)$, therefore, our task is to prove that the map $(x, y) \mapsto \inf\{|x_0, \ldots, x_k| : \{x_0, \ldots, x_k\} \in P(x, y)\}$ is a metric on $X$.

It is plain that $d(x, x) = 0$ and $d(x, y) = d(y, x)$ for every $x, y \in X$. Moreover, if $x$, $y$ and $z$ are any points in $X$, and $\{x_0, \ldots, x_k\} \in P(x, y)$ and $\{y_0, \ldots, y_l\} \in P(y, z)$, then $\{x_0, \ldots, x_k, y_1, \ldots, y_l\} \in P(x, z)$, and hence

$$d(x, z) \leq |x_0, \ldots, x_k| + |y_0, \ldots, y_l|.$$
Case 2: \( f(x_{k-1}, x_k) \geq \frac{1}{2} |x_0, ..., x_k| \). The argument in this case is identical to one given for Case 1.

Case 3: \( f(x_0, x_l) < \frac{1}{2} |x_0, ..., x_k| \) and \( f(x_{k-1}, x_k) < \frac{1}{2} |x_0, ..., x_k| \). In this case, we have \(|x_0, ..., x_k| > 0\). Let \( l \) be the largest positive integer such that \(|x_0, ..., x_l| \leq \frac{1}{2} |x_0, ..., x_k|\). (The first of our case hypotheses ensures that there is such an \( l \).) By the second of our case hypotheses, then,

\[
|x_0, ..., x_{k-1}| = |x_0, ..., x_k| - f(x_{k-1}, x_k) > \frac{1}{2} |x_0, ..., x_k|.
\]

Thus: \( l \in \{1, ..., k - 2\} \). Moreover, by the maximality of \( l \), we have \(|x_0, ..., x_{l+1}| > \frac{1}{2} |x_0, ..., x_k|\), and hence, \(|x_{l+1}, ..., x_k| < \frac{1}{2} |x_0, ..., x_k|\). So, by our induction hypothesis,

\[
\frac{1}{2} f(x_{l+1}, x_k) \leq |x_{l+1}, ..., x_k| < \frac{1}{2} |x_0, ..., x_k|,
\]

that is, \( f(x_{l+1}, x_k) < |x_0, ..., x_k| \). On the other hand, again by our induction hypothesis and the choice of \( l \),

\[
\frac{1}{2} f(x_0, x_l) \leq |x_0, ..., x_l| \leq \frac{1}{2} |x_0, ..., x_k|,
\]

so \( f(x_0, x_l) \leq |x_0, ..., x_k| \), while, obviously, \( f(x_l, x_{l+1}) \leq |x_0, ..., x_k| \). By the quadrilateral inequality, therefore, \( f(x_0, x_k) \leq 2 |x_0, ..., x_k| \), and our proof is complete.

We will use Theorem 1.1 in Chapter 12 to prove a major theorem of topological algebra.

2 Metric Subspaces

Let \( X \) be a metric space (with metric \( d \)), and \( Y \) a nonempty subset of \( X \). It is plain that \( d|_{Y \times Y} \), the restriction of \( d \) to \( Y \times Y \), is a metric on \( Y \), that is, \( (Y, d|_{Y \times Y}) \) is a metric space. We refer to this space as a **metric subspace** of \( X \), and denote it simply by \( Y \). For instance, we think of any interval, say \([0, 1]\), as a metric subspace of \( \mathbb{R} \); this means simply that the distance between any two elements \( x \) and \( y \) of \([0, 1]\) is calculated by viewing \( x \) and \( y \) as points in \( \mathbb{R} \).

**Convention.** When we consider a nonempty subset of a Euclidean space \( \mathbb{R}^n \) as a metric space without explicitly mentioning a particular metric, you should understand that we view that set as a metric subspace of \( \mathbb{R}^n \).

**Example 2.1.** For any positive integer \( n \), we think of \( \mathbb{R}^n \times \{0\} \) as a metric subspace of \( \mathbb{R}^{n+1} \). We may then "identify" \( \mathbb{R}^n \) with \( \mathbb{R}^n \times \{0\} \) in the obvious way – but we will be more formal about this in a little while – and hence think of \( \mathbb{R}^n \) as a metric subspace of \( \mathbb{R}^{n+1} \) (even though, formally speaking, \( \mathbb{R}^n \) is not a subset of \( \mathbb{R}^{n+1} \)).

**Example 2.2.** Let \( n \) be a positive integer. The set of all \( n \times n \) (real) matrices is denoted as \( \mathbb{R}^{n \times n} \). We view this set as a metric space by identifying it with the Euclidean space \( \mathbb{R}^{n^2} \). Put differently, unless otherwise is explicitly stated, we take the distance between any two \( n \times n \) (real) matrices \( A \) and \( B \) in this
text as the Euclidean distance between the \(n^2\)-vectors \((a_{11}, a_{12}, \ldots, a_{nn})\) and \((b_{11}, b_{12}, \ldots, b_{nn})\). Similarly, by the **norm** of \(A\), we understand the real number

\[
\|A\|_2 := \left( \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}^2 \right)^{1/2}.
\]

There are some metric subspaces of \(\mathbb{R}^{n \times n}\) which plays an essential role in linear algebra, as well as in other fields (such as topological groups). Of major importance is the set of all \(n \times n\) (real) invertible matrices. This set, which we view as a metric subspace of \(\mathbb{R}^{n \times n}\), is denoted as \(GL(n)\) (or \(GL(n, \mathbb{R})\)), and is called the **general linear group of degree** \(n\). (Relative to matrix multiplication, this set is indeed a group.)

\(GL(n)\) itself has some interesting metric subspaces. Of particular note is \(O(n)\), the set of all \(n \times n\) orthogonal matrices, that is, \(O(n) := \{A \in \mathbb{R}^{n \times n} : A^T A = I_n\}\), where \(I_n\) is the \(n \times n\) identity matrix.\(^5\) From linear algebra, we know that the left-inverse of a square matrix is its inverse, so every \(A \in O(n)\) is invertible (with \(A^{-1} = A^T\)). Therefore, we may think of \(O(n)\), which is called the **orthogonal group of degree** \(n\), as a metric subspace of \(GL(n)\).

Finally, recall that the determinant of any orthogonal matrix \(A\) is either \(-1\) or \(1\) (because \(\det(A)^2 = \det(A^T)\det(A) = \det(A^T A) = 1\)). The set of all \(n \times n\) orthogonal matrices with determinant \(1\) is called the **special orthogonal group of degree** \(n\), and is denoted by \(SO(n)\). (We assume here that you are familiar with the determinant function. At any rate, the definition for it is given in Example 1.14 of Chapter 2.) This set is viewed as a metric subspace of \(O(n)\), hence of \(GL(n)\), and hence of \(\mathbb{R}^{n \times n}\).\(^6\)

**Example 2.3.** Let \(c\) denote the set of all convergent real sequences. As every convergent real sequence is bounded, this set is a subset of \(\ell_\infty\). We may thus regard \(c\) as a metric subspace of \(\ell_\infty\), thereby assessing the distance between any two convergent real sequences \(x\) and \(y\) as \(\sup_{m \in \mathbb{N}} |x_m - y_m|\). Moreover, \(c\) itself has some interesting subsets. Of note are \(c_0\), the set of all real sequences that converge to \(0\), and \(c_{00}\), the set of all real sequences all but finitely many terms of which are \(0\). In what follows we will always regard either of these sets as metric subspaces of \(c\).

**Example 2.4.** Take any real numbers with \(a < b\), and recall that every continuous real function on \([a, b]\) is bounded. Thus, where \(C[a, b]\) stands for the set of all continuous real maps on \([a, b]\), we have \(C[a, b] \subseteq B[a, b]\), so we may consider \(C[a, b]\) as a metric subspace of \(B[a, b]\). We will do so in this text, thereby assessing the distance between any \(f\) and \(g\) in \(C[a, b]\) as \(\|f - g\|_\infty\), unless otherwise is explicitly mentioned.

\(^5\)You may recall from linear algebra that the members of \(O(2)\) and \(O(3)\) may be thought of as the set of all angle-preserving linear self-maps on \(\mathbb{R}^2\) and \(\mathbb{R}^3\), respectively. (Such maps are thus either rotations or reflections.)

\(^6\)We think of the members of \(SO(2)\) and \(SO(3)\) as the set of all orientation- and angle-preserving linear self-maps on \(\mathbb{R}^2\) and \(\mathbb{R}^3\), respectively. (Any one such transformation is a rotation; for this reason \(SO(3)\) is sometimes referred to as the rotation group.)
3 Products of Metric Spaces

3.1 Finite Products of Metric Spaces

Take any integer \( n \geq 2 \), and let \((X_i, d_i)\) be a metric space, \( i = 1, ..., n \). We wish to make the \( n \)-fold product \( X := X_1 \times \cdots \times X_n \) a metric space in a way that is consistent with the metric \( d_1 \) in the sense that the metric imposed on \( X \) agrees with \( d_1 \) on any set of the form \( S_1 \times \{x_2\} \times \cdots \times \{x_n\} \), where \( S_1 \) is a subset of \( X_1 \), and \( x_j \) is a point in \( X_j \) for each \( j = 2, ..., n \). There are many metrics on \( X \) that satisfy this consistency condition. For instance, the following would do nicely:

\[
\rho((x_1, ..., x_n), (y_1, ..., y_n)) := \sum_{i=1}^{n} d_i(x_i, y_i). \tag{5}
\]

Or we can let

\[
\rho((x_1, ..., x_n), (y_1, ..., y_n)) := \left( \sum_{i=1}^{n} (d_i(x_i, y_i))^p \right)^{1/p} \tag{6}
\]

for an arbitrarily fixed real number \( p \geq 1 \), or choose

\[
\rho((x_1, ..., x_n), (y_1, ..., y_n)) := \max_{i=1, ..., n} d_i(x_i, y_i). \tag{7}
\]

(Any one of these define \( \rho \) as a metric on \( X \). Why?) In fact, for most practical purposes (and certainly all purposes of this text), it is simply a matter of convenience to choose among them. (More on this point in Section 6.) Due to its simplicity, we designate (5) as our default choice.

**Definition.** Let \( n \) be a positive integer, and take \( n \) many metric spaces \((X_1, d_1), \ldots, (X_n, d_n)\). Let \( X := X_1 \times \cdots \times X_n \). We call the map \( \rho : X \times X \to \mathbb{R}^+ \) defined by (5) the **product metric** on \( X \), and refer to \((X, \rho)\) as the **product** of the metric spaces \((X_1, d_1), \ldots, (X_n, d_n)\). We may denote this space as \( X^n(X_1, d_1) \), and refer to \((X_i, d_i)\) as the \( i \)th **coordinate space** of \( X^n(X_1, d_1) \).

**Example 3.1.** \( X^n(\mathbb{R}, d_1) = \mathbb{R}^n \).

3.2 Countable Products of Metric Spaces

Life gets slightly more complicated when one is interested in the product of countably infinitely many metric spaces \((X_1, d_1), (X_2, d_2), \ldots \), for none of the alternatives (5), (6) or (7) we considered above have straightforward extensions to this case, due to obvious summability problems. (For instance, \((x_m), (y_m)\) \to \( \sum_{i=1}^{\infty} |x_i - y_i| \) is not a real-valued function on \( \mathbb{R}^\infty \).) Indeed, metrizing \( X_1 \times X_2 \times \cdots \) in a way that satisfies the consistency property proposed in the previous subsection is not possible in general. But we can still satisfy a slightly weaker consistency condition by using the following definition.
Definition. Let \((X_m, d_m)\) be a sequence of metric spaces, and put \(X := X_1 \times X_2 \times \cdots\). We call the map \(\rho : X \times X \to \mathbb{R}_+\) defined by
\[
\rho(x, y) := \sum_{i=1}^{\infty} 2^{-i} \min\{1, d_i(x, y_i)\}
\]
the product metric on \(X\), and refer to \((X, \rho)\) as the product of the metric spaces \((X_1, d_1), (X_2, d_2), \ldots\). We may denote this space as \(X^\infty(X_i, d_i)\), and refer to \((X_i, d_i)\) as the \(i\)th coordinate space of \(X^\infty(X_i, d_i)\).

Convention. In this course whenever we speak of the set \(\mathbb{R}^\infty\) of all real sequences as a metric space without mentioning a particular distance function, it is implicitly understood that \(\mathbb{R}^\infty\) is metrized by the product metric. Put differently, unless it is explicitly stated otherwise, we identify \(\mathbb{R}^\infty\) with \(X^\infty(\mathbb{R}, d_1)\).

Let \((X_m, d_m)\) be a sequence of metric spaces, and put \(X := X_1 \times X_2 \times \cdots\). Then, the map \(\rho\) we defined above on \(X \times X\) is indeed real-valued, because for any \(x, y \in X\), the sequence \(\left(\sum_{i=1}^{\infty} 2^{-i} \min\{1, d_i(x, y_i)\}\right)\) converges, being an increasing real sequence which is bounded above (by 1). The fact that \(\rho\) is a distance function is also readily verified, so calling it the “product metric” is justified.

Our definition of the product metric may seem a bit contrived at first. One might even argue that it is unnatural in the following way. “We can think of the product of finitely many metric spaces as a special case of the product of countably infinitely many metric spaces. Indeed, if \((X_1, d_1), \ldots, (X_n, d_n)\) are metric spaces, then surely, from the perspective of metric analysis, there is no discernible difference between \(X^n(X_i, d_i)\) and the product of the metric spaces \((X_1, d_1), (X_2, d_2), \ldots,\) where \(X_{n+1}, X_{n+2}, \ldots\) are all singleton sets (say, \(\{0\} = X_{n+1} = X_{n+2} = \cdots\)). But it seems like the metrization above does not accord with this point of view. After all, the way we have defined things, the product distance between \(x := (x_1, \ldots, x_n)\) and \(y := (y_1, \ldots, y_n)\) in \(X^n(X_i, d_i)\) is \(\sum_{i=1}^{n} d_i(x_i, y_i)\) whereas the product distance between \(x := (x_1, \ldots, x_n, 0, 0, \ldots)\) and \(y := (y_1, \ldots, y_n, 0, 0)\) in \(X^\infty(X_i, d_i)\) (where \(\{0\} = X_{n+1} = X_{n+2} = \cdots\)) is \(\sum_{i=1}^{n} 2^{-i} \min\{1, d_i(x_i, y_i)\}\). Is this not a problem?” It is not! Our definition of the product metric in the finite and countably infinite case are consistent with each other from a variety of viewpoints. This issue will be clarified in Section 6 below. We will also see in this and the next chapter that the above metrization of \(X_1 \times X_2 \times \cdots\) is actually a very useful one.

Remark. A subset \(S\) of a metric space \(X\) is called bounded (in \(X\)) if we can find a real number \(K > 0\) and an \(x \in X\) such that every point of \(S\) is at most \(K\) away from \(x\) (with respect to the metric of \(X\)). If \(S\) is not bounded, then it is said to be unbounded.

It is plain that the product of finitely many metric spaces is unbounded if (and only if) at least one of those metric spaces is unbounded. However, our metrization of the product of any given countably infinitely many metric spaces
always yields a bounded metric space. (Indeed, the product metric \( \rho \) takes values between 0 and 1. Thus, any two points in the product of countably many metric spaces is at most 1 away from each other.)

**Exercise**

3.1. Let \( \rho \) denote the product metric on \( \mathbb{R}^\infty \), and let \( \mathbf{0} \) stand for the sequence \((0,0,...)\). Show that \( \rho(\cdot, \mathbf{0}) \) is not a norm on \( \mathbb{R}^\infty \).

## 4 Open and Closed Sets

### 4.1 Open and Closed Balls

Let \( X \) be a metric space. For any \( x \in X \) and \( \varepsilon > 0 \), we define the **open \( \varepsilon \)-ball around** \( x \) (in \( X \)) as the set

\[
B(x, \varepsilon) := \{ \omega \in X : d(x, \omega) < \varepsilon \},
\]

and the **closed \( \varepsilon \)-ball around** \( x \) (in \( X \)) as the set

\[
\overline{B}(x, \varepsilon) := \{ \omega \in X : d(x, \omega) \leq \varepsilon \}.
\]

(Figure 1.2 illustrates how these sets look like in \( \mathbb{R}^2 \).)

![Figure 1.2](image)

The open (or closed) \( \varepsilon \)-ball around \( x \) in a metric space is never empty, for it contains \( x \). Besides, this notion is based on exactly *four* primitives. Obviously, the open (or closed) \( \varepsilon \)-ball around \( x \) in a metric space \( X \) depends on \( \varepsilon \) and \( x \). But it also depends on the set \( X \) and the distance function \( d \) used to metrize this set. The following examples are meant to illustrate this point.

**Example 4.1.** In the context of \( \mathbb{R} \), we have \( B(x, \varepsilon) = (x-\varepsilon, x+\varepsilon) \) and \( \overline{B}(x, \varepsilon) = [x-\varepsilon, x+\varepsilon] \) for any real numbers \( x \) and \( \varepsilon > 0 \). So, in \( \mathbb{R} \), we have \( B(0, \frac{1}{2}) = (-\frac{1}{2}, \frac{1}{2}) \) and \( \overline{B}(0, \frac{1}{2}) = [-\frac{1}{2}, \frac{1}{2}] \). But, in \([0,1]\), we have \( B(0, \frac{1}{2}) = [0, \frac{1}{2}) \) and \( \overline{B}(0, \frac{1}{2}) = [0, \frac{1}{2}] \).
Example 4.2. The open 1-ball around \( 0 := (0,0) \) in \( \mathbb{R}^2 \) is \( \{ (x_1,x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 < 1 \} \), whereas the open 1-ball around \( 0 \) in \( \mathbb{R} \times \{0\} \) (where we view \( \mathbb{R} \times \{0\} \) as a metric subspace of \( \mathbb{R}^2 \)) is \( \{ (x_1,0) \in \mathbb{R}^2 : -1 < x_1 < 1 \} \). Similarly, the open 1-ball of \( 0 \) in \( \mathbb{R}^2 \) is distinct from that in \( \mathbb{R}^2_p \) for \( p \neq 2 \).

As another example, we note that in Figure 1.3 all continuous real maps on \([0,1]\) whose graphs are contained in the shaded region constitute the open \( \varepsilon \)-ball around \( f \) in \( C[0,1] \).

As another example, we note that in Figure 1.3 all continuous real maps on \([0,1]\) whose graphs are contained in the shaded region constitute the open \( \varepsilon \)-ball around \( f \) in \( C[0,1] \).

![Figure 1.3](image)

\[ B(f,\varepsilon) = \{ \text{all continuous real maps on } [0,1] \text{ whose graphs are contained in the shaded region} \} \]

Notation. When the metric space under consideration is clear from the context, we will use the notation \( B(x,\varepsilon) \) to denote the \( \varepsilon \)-ball around \( x \) in that space. But when we are dealing with, say, two metric spaces \( X \) and \( Y \), we may write \( B_X(x,\varepsilon) \) and \( B_Y(y,\delta) \) for the open \( \varepsilon \)-ball around \( x \) in \( X \) and the open \( \delta \)-ball around \( y \) in \( Y \), respectively. For instance, \( B_\mathbb{R}(0,\frac{1}{2}) = (-\frac{1}{2},\frac{1}{2}) \) and \( B_{[0,1]}(0,\frac{1}{2}) = [0,\frac{1}{2}] \). In fact, later there will be a few instances in which we will consider more than one metric on the same ground set \( X \). In those situations, we will use the cumbersome, but unambiguous, notation \( B_{(X,d)}(x,\varepsilon) \) to denote the \( \varepsilon \)-ball around \( x \) in \( X \) with respect to the metric \( d \). (Similar notational conventions apply to closed balls as well.)

4.2 Diameter

The diameter of a nonempty set \( S \) in a metric space \( X \) is defined as the extended real number \( \sup \{ d(x,y) : x,y \in S \} \); we denote this quantity by \( \text{diam}(S) \). (As such, \( \text{diam} \) can be thought of as a real map on \( 2^X \).) See Figure 1.4.

An immediate application of the triangle inequality shows that

\[ \text{diam}(B(x,\varepsilon)) \leq 2\varepsilon \]

for any \( x \in X \) and \( \varepsilon > 0 \). (It follows that \( S \) is bounded iff \( \text{diam}(S) < \infty \).) But the equality need not hold here. For instance, the diameter of \( B_{[0,1]}(0,\frac{1}{2}) \) is \( \frac{1}{2} \). Nevertheless, diameter is a monotonic set function, that is, \( S \subseteq T \) implies
diam(S) ≤ diam(T). But diam(S) = diam(T) may hold when S is a proper subset of T. (For instance, diam(0, 1) = 1 = diam[0, 1]. In fact, in \( \mathbb{R}^n \) (but not in an arbitrary metric space), we always have \( \text{diam}(B(x, \varepsilon)) = \text{diam}(\overline{B}(x, \varepsilon)) \).)

### 4.3 Open and Closed Sets

Open balls play a major role in mathematical analysis mainly through the following definition.

**Definition.** Let X be a metric space. We say that a subset S of X is open in X (or that it is an **open subset of** X) if for each \( x \in S \), there is an \( \varepsilon > 0 \) such that \( B(x, \varepsilon) \subseteq S \). We say that S is closed in X (or that it is a **closed subset of** X) if \( X \setminus S \) is open in X.

**Notation.** We denote the set of all open subsets of a metric space X by \( \mathcal{O}_X \). (This collection is called the **topology** of X, but we will refrain from using this term until we get to Chapter 4.) In turn, we denote the set of all closed subsets of X by \( \mathcal{C}_X \).

**Historical Note.** The notions of open and closed sets, as well as many related topics (such as the interior and closure of a set, limit points, etc.) were already developed as early as 1872 by the legendary mathematician Georg Cantor, commonly known as the “father of set theory.” But Cantor worked exclusively within the confines of \( \mathbb{R} \) (with only occasional extensions to \( \mathbb{R}^n \)). The formulation, and further investigation, of these concepts in the context of metric spaces were subsequently carried out by numerous researchers, beginning primarily by Fréchet’s 1906 contribution.

Because open balls around a point are inherently linked to the underlying metric space, so do the open and closed sets. Always keep in mind that changing the metric on a given set, or concentrating on a metric subspace of the original metric space, is likely to yield different classes of open (and hence closed) sets.

**Example 4.3.** In any metric space X, the sets X and \( \emptyset \) are both open and closed. (We refer to the sets which are both open and closed as **clopen**.)

**Example 4.4.** Let X be a metric space. For any \( x \in X \) and \( \varepsilon > 0 \), the set \( B(x, \varepsilon) \) is open, while \( \overline{B}(x, \varepsilon) \) is closed. To prove the first assertion, take any
$y \in B(x, \varepsilon)$ and define $\delta := \varepsilon - d(x, y) > 0$. We have $B(y, \delta) \subseteq B(x, \varepsilon)$ because, the triangle inequality implies that, for any $z \in B(y, \delta)$, we have

$$d(x, z) \leq d(x, y) + d(y, z) < d(x, y) + \varepsilon - d(x, y) = \varepsilon.$$  

(See Figure 1.5 to see the intuition of the argument.) As $y$ is arbitrary in $B(x, \varepsilon)$ here, we may thus conclude that $B(x, \varepsilon)$ is open. We leave proving that $\overline{B}(x, \varepsilon)$ is closed as an exercise. (Hint: Put $O := X \setminus \overline{B}(x, \varepsilon)$, pick any $y \in O$ and $\delta \in (0, d(x, y) - \varepsilon)$, and use the reverse triangle inequality to find $B(y, \delta) \subseteq O$.)

![Figure 1.5](image-url)

**Example 4.5.** Let $X$ be a metric space. For any $x \in X$, the singleton $\{x\}$ is closed. To prove this, we need to show that $X \setminus \{x\}$ is open. If $X = \{x\}$, there is nothing to prove. (Yes?) On the other hand, if there is a $y \in X \setminus \{x\}$, we have $B(y, d(x, y)) \subseteq X \setminus \{x\}$. It follows that $X \setminus \{x\}$ is open, and $\{x\}$ is closed.

**Example 4.6.** Any subset $S$ of a nonempty set $X$ is open with respect to the discrete metric. For, if $x \in S \subseteq X$, then $B(x, \frac{1}{2}) = \{x\} \subseteq S$, where the discrete metric is used in computing $B(x, \frac{1}{2})$. Thus: *Any subset of a discrete metric space is clopen.* (In other words, for a discrete metric space $X$, we have $O_X = 2^X = C_X$.)

**Example 4.7.** It is possible for a set in a metric space to be neither open nor closed. In $\mathbb{R}$, for instance, $(0, 1)$ is open, $[0, 1]$ is closed, and $[0, 1)$ is neither open nor closed. But observe that the structure of the mother metric space is crucial for the validity of these statements. For instance, the set $[0, 1)$ is open when considered as a set in the metric space $\mathbb{R}_+$.  

**Example 4.8.** For any positive integer $n$, let $X$ be the product of the metric spaces $(X_1, d_1), ..., (X_n, d_n)$. Then, for any $O \in O_X$ and any $(x_1, ..., x_n) \in O$, there is an $\varepsilon > 0$ such that

$$B_{X_1}(x_1, \varepsilon) \times \cdots \times B_{X_n}(x_n, \varepsilon) \subseteq O.$$  

(8)
Given that $O$ is open in $X$, there is a $\delta > 0$ such that the open $\delta$-ball around $x$ in $X$ is contained in $O$, that is, for any $(y_1, \ldots, y_n)$ in $X$,

$$\sum_{i=1}^{n} d_i(x_i, y_i) < \delta \quad \text{implies} \quad (y_1, \ldots, y_n) \in O,$$

and hence, for $\varepsilon := \delta/n$, (8) holds.) Therefore, every open set $O$ in $X$ can be written as the union of all sets of the form $B_{X_1}(x_1, \varepsilon) \times \cdots \times B_{X_n}(x_n, \varepsilon)$ where $(x_1, \ldots, x_n) \in O$, and $\varepsilon > 0$ is chosen so that (8) holds.

**Example 4.9.** Let us redo the previous example, this time using countably infinitely many metric spaces $(X_1, d_1), (X_2, d_2), \ldots$. Let $X$ be the product of these metric spaces. Then, we claim, for any $O \in O_X$ and $(x_1, x_2, \ldots) \in O$, there is an $\varepsilon > 0$ and a positive integer $n$ such that

$$B_{X_1}(x_1, \varepsilon) \times \cdots \times B_{X_n}(x_n, \varepsilon) \times X_{n+1} \times X_{n+2} \times \cdots \subseteq O.$$  \hspace{1cm} (9)

Indeed, since $O$ is open in $X$, there is a $\delta > 0$ such that the open $\delta$-ball around $x$ in $X$ is contained in $O$, that is, for any $(y_1, y_2, \ldots)$ in $X$,

$$\sum_{i=1}^{\infty} 2^{-i} \min\{1, d_i(x_i, y_i)\} < \delta \quad \text{implies} \quad (y_1, y_2, \ldots) \in O.$$ \hspace{1cm} (10)

Let us then choose $n \in \mathbb{N}$ large enough that $\sum_{i=n+1}^{\infty} 2^{-i} < \delta/2$, and set $\varepsilon := \min\{1, \delta/2n\}$. Then, for any $(y_1, y_2, \ldots)$ in the left-hand side of (9), we have

$$\sum_{i=1}^{\infty} 2^{-i} \min\{1, d_i(x_i, y_i)\} < \sum_{i=1}^{n} \frac{\delta}{2^n} + \frac{\delta}{2} = \delta,$$

and hence (10) entails that $(y_1, y_2, \ldots) \in O$, thereby proving (9). Again, as a consequence of this fact, we find that every open set $O$ in $X$ can be written as the union of all sets of the form $B_{X_1}(x_1, \varepsilon) \times \cdots \times B_{X_n}(x_n, \varepsilon) \times X_{n+1} \times X_{n+2} \times \cdots$, where $(x_1, x_2, \ldots) \in O$, and $\varepsilon > 0$ and $n \in \mathbb{N}$ are chosen so that (8) holds.

**Remark.** Example 4.9 says that any open set in the product of countably infinitely many metric spaces includes sets like those in the left-hand side of (9). This may seem odd, as it forces the open sets in such product spaces to be rather large. In particular, a set like $B_{X_1}(x_1, \varepsilon) \times B_{X_n}(x_n, \varepsilon) \times \cdots$ need not be open with respect to the product metric.

To drive this point home, let us consider $\mathbb{R}^\infty$ which we view as $X^\infty(\mathbb{R}, d_1)$. Then, contrary to what one may at first presume, $S := (0,1) \times (0,1) \times \cdots$ is not open in $\mathbb{R}^\infty$. The argument given in Example 4.9 actually proves this, but if only to fix ideas, let us give a direct proof for it. Let $x := (\frac{1}{2}, \frac{1}{2}, \ldots)$, which obviously belongs to $S$. If $S$ were open, there would exist an $\varepsilon > 0$ such that the open $\varepsilon$-ball around $x$ fits within $S$. But, if we choose $n \in \mathbb{N}$ large enough so that $\sum_{i=n+1}^{\infty} 2^{-i} < \varepsilon/2$, every real sequence $y := (y_m)$ with
\[
\sum_{i=1}^{n} \left| \frac{1}{2} - y_i \right| \leq \varepsilon/2 \text{ satisfies } \rho(x, y) < \varepsilon, \text{ where } \rho \text{ is the product metric on } \mathbb{R}^\infty. \text{ But of course, such a sequence need not reside in } S. (\text{For instance, the } \rho \text{ distance between } (\frac{a}{2\alpha}, ..., \frac{a}{2\alpha}, 10, 10, ...) \text{ and } x \text{ are strictly less than } \varepsilon, \text{ but obviously, this sequence does not belong to } S. ) \text{ Conclusion: } S \text{ is not open in } \mathbb{R}^\infty.
\]

Any Union, as well as Finite Intersection, of Open Sets is Open

Let us now look at how the notion of “openness” behave with respect to taking unions and intersections.

**Proposition 4.1.** The union of any nonempty collection of open sets in a metric space is open. Moreover, the intersection of any nonempty finite collection of open sets in a metric space is open.

**Proof.** The proof of the first assertion is particularly easy; we leave this as an exercise. To prove the second assertion, let \( m \) be a positive integer and \( O_1, ..., O_m \) open sets in a given metric space \( X \). Put \( O := O_1 \cap \cdots \cap O_m \), and take any \( x \in O \). Our task is to find an \( \varepsilon > 0 \) such that \( B(x, \varepsilon) \subseteq O \). To this end, notice that \( x \in O_i \) for each \( i \), and hence, as \( O_i \) is open, there is an \( \varepsilon_i > 0 \) with \( B(x, \varepsilon_i) \subseteq O_i \). Then, where \( \varepsilon := \min\{\varepsilon_1, ..., \varepsilon_m\} \), we have \( B(x, \varepsilon) \subseteq O \), and we are done.

**Warning.** The intersection of infinitely many open sets in a metric space need not be open. For instance, \((-1/m, 1/m) \in O_k \) for each positive integer \( m \), but the intersection of these sets is \( \{0\} \) which does not belong to \( O_k \).

**Example 4.10.** Take any integer \( n \geq 2 \). Let \((X_1, d_1), ..., (X_n, d_n)\) be metric spaces, and \((X, \rho)\) the product of these spaces. If \( O_i \) is an open set in \( X_i \) for each \( i \), then \( O := O_1 \times \cdots \times O_n \) is an open subset of \( X \). (To see this, take any \( x := (x_1, ..., x_n) \) in \( O \). We are given that for each \( i \) there is an \( \varepsilon_i > 0 \) such that \( B(x_i, \varepsilon_i) \subseteq O_i \). Then, putting \( \varepsilon := \min\{\varepsilon_1, ..., \varepsilon_n\} \), we find \( B(x, \varepsilon) \subseteq O \). (Check!)) Similarly, if \( C_i \in C_{X_i} \) for each \( i \), then \( C := C_1 \times \cdots \times C_n \in C_X \). Indeed,

\[
X \setminus C = [(X_1 \setminus C_1) \times X_2 \times \cdots \times X_n] \cup \cdots \cup [X_1 \times \cdots \times X_{n-1} \times (X_n \setminus C_n)],
\]

and it follows from what we have just found and Proposition 4.1 that \( X \setminus C \) is an open set in \( X \).

By taking complements and using De Morgan’s Laws, we can formulate Proposition 4.1 for closed sets in the following manner:

**Proposition 4.2.** The intersection of any nonempty collection of closed sets in a metric space is closed. Moreover, the union of any nonempty finite collection of closed sets in a metric space is closed.

**Warning.** The union of infinitely many closed sets in a metric space need not be closed. (Exercise: Give an example to show this.)
Example 4.11. We have seen in Example 4.5 that every singleton set in a metric space is closed. In view of the second part of Proposition 4.2, therefore, we may conclude: *Every finite subset of a metric space is closed.*

Open Sets in Metric Subspaces

Let $X$ be a metric space and $Y$ a metric subspace of $X$. Take a subset $S$ of $Y$. As $S$ is also a subset of $X$, when we refer to $S$ as an “open set” (or a “closed set”), there is an ambiguity in the statement. Instead, we should be clear about relative to which space we declare the “openness” (or “closedness”) of the set $S$. Consequently, we will refer to the set $S$ as “open in $Y$” (or “closed in $Y$”) if it is $Y$ that we consider as the underlying metric space. Some authors use the terminology “relatively open in $Y$” (or “relatively closed in $Y$”) for this purpose.

Example 4.7 shows that a set that is open (or closed) in a metric subspace need not be open (or closed) in the mother metric space. But that does not mean that there is no connection between those sets that are open in the mother space and in a given metric subspace. The following, very useful, result clarifies that matter. (Part (a) of this result is illustrated in Figure 1.6.)

![Figure 1.6](image_url)

**Proposition 4.3.** Let $X$ be a metric space and $Y$ a metric subspace of $X$. Let $S$ be a subset of $Y$. Then,

(a) $S$ is open in $Y$ iff there is an open subset $O$ of $X$ with $S = O \cap Y$;

(b) $S$ is closed in $Y$ iff there is a closed subset $C$ of $X$ with $S = C \cap Y$.

**Proof.** It is enough to prove (a), because (b) follows from (a) by taking complements and using De Morgan’s Laws. To prove (a), suppose first that $S$ is open in $Y$. Then, for any $x \in S$ there is an $\varepsilon_x > 0$ such that $B_Y(x, \varepsilon_x) \subseteq S$; hence, $S = \bigcup_{x \in S} B_Y(x, \varepsilon_x)$. Since $B_Y(x, \varepsilon_x) = B(x, \varepsilon_x) \cap Y$ for each $x$, therefore,

$$S = \bigcup_{x \in S} (B(x, \varepsilon_x) \cap Y) = \left( \bigcup_{x \in S} B(x, \varepsilon_x) \right) \cap Y.$$  

Putting $O := \bigcup_{x \in S} B(x, \varepsilon_x)$, and noting that $O$ is open in $X$ (Proposition 4.1), proves the “only if” part of our assertion. Conversely, take any $O \in \mathcal{O}_X$ with
Then, for any \( x \in S \), there is an \( \varepsilon > 0 \) with \( B(x, \varepsilon) \subseteq O \), and hence \( B_Y(x, \varepsilon) = B(x, \varepsilon) \cap Y \subseteq O \cap Y = S \). This shows that \( S \) is open in \( Y \).

**Warning.** Given any metric space \( X \), let \( Y \) be a metric subspace of \( X \), and take any \( U \subseteq Y \). An immediate application of Proposition 4.3 shows that

\[
U \in \mathcal{O}_X \quad \text{implies} \quad U \in \mathcal{O}_Y.
\]

(Yes?) But the converse is false! For instance, \([0,1)\) is open in \( \mathbb{R}_+ \), but not in \( \mathbb{R} \). Yet, if the metric subspace under consideration is an open subset of the mother metric space, all goes well. Put precisely, *provided that \( Y \) is open in \( X \),

\[
U \text{ is open in } X \iff U \text{ is open in } Y.
\]

(*) Proof. If \( U \in \mathcal{O}_Y \), then by Proposition 4.3, there is an \( O \in \mathcal{O}_X \) such that \( U = O \cap Y \). So, if \( Y \) is open in \( X \) as well, \( U \) must be open in \( X \), being the intersection of two open sets in \( X \).)

Similar remarks apply to closed sets as well, of course. If \( C \subseteq Y \), then

\[
C \text{ is closed in } X \quad \text{only if} \quad C \text{ is closed in } Y,
\]

and, *provided that \( Y \) is closed in \( X \),

\[
C \text{ is closed in } X \iff C \text{ is closed in } Y.
\]

### 4.4 \( G_\delta \) and \( F_\sigma \) Sets

We say that a subset \( S \) of a metric space \( X \) is a \( G_\delta \)-subset of that space if \( S \) can be expressed as the intersection of countably many open subsets of \( X \). (Here \( G \) comes from the German word *gebiet* for open set (or neighborhood) and \( \delta \) from the German word *Durchschnitt* for intersection.) For instance, \( \mathbb{R} \setminus \mathbb{Q} \) is a \( G_\delta \)-subset of \( \mathbb{R} \) because this set equals \( \bigcap \{ \mathbb{R} \setminus \{ r \} : r \in \mathbb{Q} \} \). In the next chapter, we will show that \( \mathbb{Q} \) is not a \( G_\delta \)-subset of \( \mathbb{R} \).

Dually, an \( F_\sigma \)-subset of \( X \) is a subset of \( X \) that equals \( \bigcup C \) for some countable \( C \subseteq C_X \). (Here \( F \) comes from the French word *fermé* for closed and \( \sigma \) from the French word *somme* for sum.) It is plain that the complement of a \( G_\delta \)-subset of \( X \) is an \( F_\sigma \)-subset of \( X \), and conversely.

\( G_\delta \) and \( F_\sigma \)-sets are quite important for real analysis and set theory. This is partly due to the fact that every closed subset of a metric space \( X \) is a \( G_\delta \), and every open subset of \( X \) is a \( F_\sigma \). (We will prove this in an exercise in the next chapter.) Such sets will at times play an essential role in this course as well.

### 4.5 Interior, Closure and Boundary

Let \( S \) be a subset of a metric space \( X \). The **interior** of \( S \) (in \( X \)) is the set defined by

\[
\text{int}(S) := \bigcup \{ O \in \mathcal{O}_X : O \subseteq S \}.
\]
This is the largest open set in $X$ that is contained in $S$. (That is, $\text{int}(S)$ is an open set that is contained in $S$, and if $O$ is another such set, then $O \subseteq \text{int}(S)$.) Similarly, the closure of $S$ (in $X$) is the set defined by

$$\text{cl}(S) := \bigcap \{ C \in \mathcal{C}_X : S \subseteq C \}.$$  

This is the smallest closed set in $X$ that contains $S$. (That is, $\text{cl}(S)$ is a closed set that contains $S$, and if $C$ is another such set, then $\text{cl}(S) \subseteq C$.) Finally, we define the boundary of $S$ (in $X$) as

$$\partial S := \text{cl}(S) \setminus \text{int}(S).$$

Figure 1.7 provides a few illustrations of these concepts in $\mathbb{R}^2$.

Warning. Let $X$ be a metric space, and $Y$ a metric subspace of $X$. For any subset $S$ of $Y$, we may think of the interior of $S$ as lying in $X$ or in $Y$. (Again, these may well be different. For instance, the interior of $[0,1]$ in $\mathbb{R}$ is $(0,1)$, but that of $[0,1]$ in $\mathbb{R}_+$ is $[0,1]$.) It is for this reason that we will sometimes use the notation $\text{int}_X(S)$, instead of $\text{int}(S)$, to mean the interior of $S$ relative to the metric space $X$. However, if there is only one metric space under consideration, or the context leaves no room for confusion, we simply write $\text{int}(S)$ to denote the interior of $S$ relative to the appropriate space. (The same comments apply to the closure and boundary operators as well.)

Remark. In the context of any metric space $X$, we have $\text{int}(\emptyset) = \emptyset = \text{cl}(\emptyset)$ and $\text{int}(X) = X = \text{cl}(X)$.

Remark. In the context of any metric space $X$, a subset $S$ of $X$ is open iff $S = \text{int}(S)$ while it is closed iff $S = \text{cl}(S)$. These facts follow readily from the definitions above.

The definitions of the interior and closure of a set are not helpful in computing these sets in practice. The following characterization is often more useful in that regard.

**Proposition 4.4.** Let $X$ be a metric space, $x \in X$, and $S$ a subset of $X$. Then,

(a) $x \in \text{int}(S)$ iff $B(x, \epsilon) \subseteq S$ for some $\epsilon > 0$;
that is contained in \( \text{diam} \) is, say, the discrete metric space, then \( \text{cl} \) may not hold here. It holds, for instance, in the context of any \( \mathbb{R} \) real numbers, we see that \( \text{int} \) Proposition 4.4 and the fact that there is an irrational number between any two of any of these intervals is \( \{a, b\} \).) The interior and closure (in \( \mathbb{R} \)) of either of the intervals \( (-\infty, a) \) and \( (-\infty, a) \) are \( (-\infty, a) \) and \( (-\infty, a) \), respectively. (The boundary of any of these intervals is \( \{a\} \).) Question: What is the closure of \( (-\infty, a) \) in \( \mathbb{R}^n \)?

Example 4.13. What are the interior and closure of \( \mathbb{Q} \) (in \( \mathbb{R} \))? Using part (a) of Proposition 4.4 and the fact that there is an irrational number between any two real numbers, we see that \( \text{int}(\mathbb{Q}) = \emptyset \). Similarly, using part (b) of Proposition 4.4 and the fact that there is a rational number between any two real numbers, we see that \( \text{cl}(\mathbb{Q}) = \mathbb{R} \). (Thus: \( \partial \mathbb{Q} = \mathbb{R} \).) Similarly, \( \text{int}(\mathbb{R} \setminus \mathbb{Q}) = \emptyset \) and \( \text{cl}(\mathbb{R} \setminus \mathbb{Q}) = \mathbb{R} \). (Things can actually get more complicated in \( \mathbb{R} \). For instance, an uncountable closed subset of \( \mathbb{R} \) may well have an empty interior; see Exercise 4.7.)

Example 4.14. Let \( X \) be a metric space, and take any \( x \in X \) and \( \varepsilon > 0 \). It is always true that

\[
\text{cl}(B(x, \varepsilon)) \subseteq \overline{B}(x, \varepsilon),
\]

because \( \overline{B}(x, \varepsilon) \) is a closed set in \( X \) that contains \( B(x, \varepsilon) \). But the equality may or may not hold here. It holds, for instance, in the context of any \( \mathbb{R}^n \). But if \( X \) is, say, the discrete metric space, then \( \text{cl}(B(x, 1)) = \{x\} \) while \( \overline{B}(x, 1) = X \). As another example, let \( X \) be the set \( (-\infty, -1] \cup \{0\} \cup [1, \infty) \), and view this set as a metric subspace of \( \mathbb{R} \). Then, \( \text{cl}(B(0, 1)) = \{0\} \) while \( \overline{B}(x, 1) = \{-1, 0, 1\} \).

Warning. Let \( X \) be a metric space, and take any \( x \in X \) and \( \varepsilon > 0 \). Then, we are sure to have \( \text{diam}(B(x, \varepsilon)) = \text{diam}(\text{cl}(B(x, \varepsilon))) \), but it may well be the case that \( \text{diam}(B(x, \varepsilon)) < \text{diam}(\overline{B}(x, \varepsilon)) \).

Exercises

4.1. Which subsets of \( \mathbb{R} \) are clopen?
4.2. Let \( X := (-\infty, -1) \cup (1, \infty), \) and view this as a metric subspace of \( \mathbb{R}. \) Which subsets of \( X \) are clopen?

4.3. For any positive integer \( k, \) let \( e_k \) denote the real sequence whose \( k \)th term is 1 and whose every other term is 0. Show that \( \{e_1, e_2, \ldots\} \) is closed in \( \ell_\infty. \) Is this true in \( \ell_1? \)

4.4. Is \( \{ f \in C[0, 1] : f > 0 \} \) an open set in \( C[0, 1]? \) In \( B[0, 1]? \)

4.5. Let \( C_b(\mathbb{R}) \) stand for the set of all continuous and bounded real self-maps on \( \mathbb{R}, \) and view this as a metric subspace of \( B(\mathbb{R}). \) Is \( \{ f \in C_b(\mathbb{R}) : f > 0 \} \) an open set in \( C_b(\mathbb{R})? \)

4.6. Consider the (ultra)metric space we have introduced in Exercise 1.8. Show that no singleton is open in this space.

4.7. (The Cantor Set) Here is a famous construction. Begin with removing the open middle third of \( [0, 1], \) namely, \( \left( \frac{1}{3}, \frac{2}{3} \right), \) to obtain the set \( C_1 := [0, \frac{1}{3}] \cup \left[ \frac{2}{3}, 1 \right]. \) Next, remove the open middle thirds of the two component subintervals of \( C_1, \) namely, \( \left( \frac{1}{9}, \frac{2}{9} \right) \) and \( \left( \frac{7}{9}, \frac{8}{9} \right). \) This leaves us with \( C_2 := [0, \frac{1}{9}] \cup \left[ \frac{2}{9}, \frac{1}{3} \right] \cup \left[ \frac{2}{3}, \frac{7}{9} \right] \cup \left[ \frac{8}{9}, 1 \right]. \) Then remove the open middle thirds of the four component subintervals of \( C_2, \) and proceed inductively. (See Figure 1.8.) This procedure yields a sequence \( (C_m) \) of subsets of \( [0, 1], \) which can be defined, recursively, as

\[
C_0 := [0, 1] \quad \text{and} \quad C_m := \frac{1}{3} C_{m-1} \cup \left( \frac{2}{3} + \frac{1}{3} C_{m-1} \right), \quad m = 1, 2, \ldots
\]

The Cantor set \( C \) is defined as the set of all points in \( [0, 1] \) not deleted throughout this procedure, that is, \( C := \bigcap_{i=0}^{\infty} C_i. \) This set is nonempty. (In fact, the endpoints of any interval that is removed in our procedure are contained in \( C. \))

a. Show that there is a one-to-one correspondence between \( \{0, 1\}^\infty \) and \( C, \) and conclude that \( C \) is uncountable.

b. Show that \( C \) is a closed set with empty interior. Moreover, no singleton subset of \( C \) is open.

4.8. Show that for every distinct points \( x \) and \( y \) in a metric space \( X, \) there exist disjoint open sets \( O \) and \( U \) in \( X \) such that \( x \in O \) and \( y \in U. \)

4.9. Show that there is no metric on \( \mathbb{N} \) that would render a nonempty subset of \( \mathbb{N} \) open iff the complement of that set is finite.
4.10. Consider the metric $d$ introduced in Exercise 1.2 where $X = \mathbb{R}^2$ and $f(x) := \|x\|_2$. (Here, of course, $\|\cdot\|_2$ stands for the Euclidean norm.) Assuming that the underlying metric space is $(\mathbb{R}^2, d)$, prove:

a. For any $x \in \mathbb{R}^2 \setminus \{0\}$ and $\varepsilon \in (0, \|x\|_2)$, the $\varepsilon$-ball around $x$ is $\{x\}$;

b. For any $x \in \mathbb{R}^2 \setminus \{0\}$ and $\varepsilon \in (\|x\|_2, \infty)$, the $\varepsilon$-ball around $x$ is $\{x\} \cup B_{\mathbb{R}^2}(0, \varepsilon - \|x\|_2)$;

c. Any subset of $\mathbb{R}^2 \setminus \{0\}$ is open;

d. If $S$ is a subset of $\mathbb{R}^2$ with $0 \in S$, then $S$ is open iff there is an open ball that is contained in $S$.

4.11. Consider the metric space $(\mathbb{N}, d)$ introduced in Exercise 1.5. Take any positive integer $i$, and compute $B(i, \varepsilon)$ for any $\varepsilon \in (0, 1 + 2^{-i})$. What does this ball look like for $\varepsilon > 1 + 2^{-i}$?

4.12. Let $X$ be an ultrametric space, and take any $x, y \in X$ and $\varepsilon > 0$. Show that if $y \in B(x, \varepsilon)$, then $B(y, \varepsilon) = B(x, \varepsilon)$. (So, an open ball in an ultrametric space may have several centers!) Also show that if $B(x, \varepsilon_1)$ and $B(y, \varepsilon_2)$ overlap for some $\varepsilon_1, \varepsilon_2 > 0$, then either $B(x, \varepsilon_1)$ is contained in $B(y, \varepsilon_2)$ or vice versa.

4.13. Let $X$ be an ultrametric space, and take any $x \in X$ and $\varepsilon > 0$. Show that $B(x, \varepsilon)$ is clopen, and conclude that $\partial B(x, \varepsilon) = \emptyset$.

4.14. Show that every open set in any $\mathbb{R}^{n,p}$ can be written as the union of countably many open balls in that space, for any $p \in [1, \infty]$.

4.15. Define $D : \mathbb{R}^\infty \times \mathbb{R}^\infty \to \mathbb{R}_+$ by $D((x_m), (y_m)) := \sup_{m \in \mathbb{N}} \min\{1, d(x_m, y_m)\}$, and check that $D$ is a metric on $\mathbb{R}^\infty$. Show that every set in $\mathbb{R}^\infty$ which is open with respect to the product metric is open with respect to $D$, and give an example to show that the converse is false.

4.16. Let $S$ be a subset of a metric space $X$. Prove:

a. $\partial S \subset C_X$, and int$(\partial S) = \emptyset$ provided that $S$ is closed;

b. $\partial \partial S \subset \partial S$, but equality need not hold;

c. $\partial \partial S = \partial \partial \partial S$;

d. $\partial S = S$ iff $S = \emptyset$;

e. $\partial S = \emptyset$ iff $S$ is clopen.

4.17. Let $S$ be a subset of a metric space $X$. Prove: $\partial S = \cl(S) \cap \cl(X \backslash S)$ and int$(S) = X \backslash \cl(X \backslash S)$.

4.18. Let $k$ be a positive integer and $S_1, \ldots, S_k$ subsets of a metric space $X$. Show that

$$\cl\left(\bigcup_{i=1}^k S_i\right) = \bigcup_{i=1}^k \cl(S_i) \quad \text{and} \quad \int\left(\bigcup_{i=1}^k S_i\right) \supseteq \bigcup_{i=1}^k \int(S_i).$$

(Warning. Equality need not hold in the second statement. For instance, in the context of $\mathbb{R}$, int$((0, 1] \cup [1, 2)) = (0, 2)$ but int$((0, 1]\cup \int([1, 2)) = (0, 2) \backslash \{1\}$.

Also show that

$$\cl\left(\bigcap_{i=1}^k S_i\right) \subseteq \bigcap_{i=1}^k \cl(S_i) \quad \text{and} \quad \int\left(\bigcap_{i=1}^k S_i\right) = \bigcap_{i=1}^k \int(S_i).$$
5 Sequences in a Metric Space

5.1 Convergent Sequences

The notion of openness (and hence closedness) of a set in a metric space can be characterized by means of the sequences that live in that space. Before we get to this characterization, we will first look at what it means for a sequence to converge in a metric space.

**Definition.** Let \( X \) be a metric space, \( x \in X \), and \((x_m) \in X^\infty\). We say that \((x_m)\) converges to \( x \) if for each \( \varepsilon > 0 \), there is a real number \( M \) (that may depend on \( \varepsilon \)) such that \( d(x_m, x) < \varepsilon \) for all \( m \geq M \). In this case, we say that \((x_m)\) converges in \( X \), or that it is convergent (in \( X \)), we refer to \( x \) as the limit of \((x_m)\), and write either \( x_m \to x \) or \( \lim x_m = x \).

Thus, a sequence \((x_m)\) in a metric space \( X \) converges to a point \( x \) in this space, if for any \( \varepsilon > 0 \), all but finitely many terms of the sequence \((x_m)\) belong to \( B(x, \varepsilon) \). One way of thinking about this intuitively is viewing the sequence \((x_m)\) as "residing in \( B(x, \varepsilon) \) eventually" no matter how small \( \varepsilon \) is. Equivalently, we have \( x_m \to x \) iff for every \( O \in \mathcal{O}_X \) with \( x \in O \), there is a positive integer \( M \) such that \( x_m \in O \) for all \( m \geq M \). So, for instance, in the context of \( \mathbb{R} \), we have \( 1/m \to 0 \), because, for any open subset \( O \) on \( \mathbb{R} \) with \( 0 \in O \), there is an \( M \in \mathbb{N} \) such that \( 1/m \in O \) for every \( m \geq M \).

**Warning.** Whether or not a sequence converges in a metric space, or where it converges, depends very much on the underlying distance function of the space. For instance, consider the map self-map \( f \) on \( \mathbb{R} \) defined by \( f(0) := 1, f(1) := 0, \) and \( f(x) := x \) for every \( x \in \mathbb{R} \setminus \{0, 1\} \). Then, the real map \( D \) on \( \mathbb{R} \times \mathbb{R} \) defined by \( D(x, y) := |f(x) - f(y)| \) is a metric on \( \mathbb{R} \). Relative to this metric, we have \( 1/m \to 1 \).

It is important to note that a sequence \((x_m)\) in a metric space can converge to at most one limit. Indeed, if we have \( x_m \to x \) and \( x_m \to y \) with distinct \( x \) and \( y \), then for any \( \varepsilon > 0 \), there exists \( M_1, M_2 \) such that \( d(x_m, x) < \varepsilon/2 \) for all \( m \geq M_1 \) and \( d(x_m, y) < \varepsilon/2 \) for all \( m \geq M_2 \). Thus, \( d(x, y) \leq d(x, x_m) + d(x_m, y) \leq \varepsilon/2 + \varepsilon/2 = \varepsilon \) for all \( m \geq \max\{M_1, M_2\} \), contradicting the uniqueness of the limit.
and $y$, then $\varepsilon := d(x, y) > 0$, and hence $x_m \in B(x, \frac{\varepsilon}{2}) \cap B(y, \frac{\varepsilon}{2})$ for some $m$. But this implies $d(x, y) \leq d(x, x_m) + d(x_m, y) < \varepsilon$, a contradiction.

**Example 5.1.** A sequence $(x_m)$ is convergent in a discrete space if and only if it is eventually constant (that is, there is an $M \in \mathbb{N}$ such that $x_M = x_{M+1} = \cdots$).

**Example 5.2.** A constant sequence in any metric space is convergent.

**Example 5.3.** Take any positive integer $n$, and let $(x_m) = ((x_{1,m}, \ldots, x_{n,m}))$ be a sequence in $\mathbb{R}^n$. It is easy to show that $x_m \to (x_1, \ldots, x_n)$ if $(x_{i,m})$ converges to $x_i$ for each $i = 1, \ldots, n$. (Exercise: Prove this in your head!)

**Example 5.4.** For each $m \in \mathbb{N}$, consider the following real sequences:

$$x_m := (0, \ldots, 0, \frac{1}{m}, 0, \ldots), \quad y_m := (0, \ldots, 0, 1, 0, \ldots), \quad z_m := (\frac{1}{m}, \ldots, \frac{1}{m}, 0, \ldots).$$

(Here the only nonzero term of the sequences $x_m$ and $y_m$ is the $m$th one, and all but the first $m$ terms of $z_m$ are zero.) Let $0$ stand for the zero sequence $(0, 0, \ldots)$. Since $d_p(x_m, 0) = \|x_m\|_p = \frac{1}{m}$ for each $m$, we have $(x_m) \to 0$ in $\ell_p$ for any $p \in [1, \infty]$. By contrast, it is easily checked that the sequence $(y_1, y_2, \ldots)$ is not convergent in $\ell_p$ for any $p \in [1, \infty]$. On the other hand, we have $\|z_m\|_\infty = \frac{1}{m} \downarrow 0$, so $(z_m)$ converges to 0 in $\ell_\infty$. Yet $d_1(z_m, 0) = \|z_m\|_1 = 1$ for each $m$, so $(z_m)$ does not converge to 0 in $\ell_1$. Question: Is $(z_m)$ convergent in any $\ell_p$ when $p \in (1, \infty)$?

**Example 5.5.** For any positive integer $m$, define $f_m \in B[0, 1]$ by $f_m(t) := t^m$. Since $f_m(t) \to 0$ for all $t \in [0, 1)$ and $f_m(1) = 1$ for all $m$, it may at first seem plausible that $f_m \to f$ where $f(t) = 0$ for all $t \in [0, 1)$ and $f(1) = 1$. But this is false, because $\|f_m - f\|_\infty = 1$ for every $m \in \mathbb{N}$. This example shows, again, how detrimental the choice of the metric may be in studying the convergence of sequences.

**Remark.** For any metric space $X$, $x \in X$ and a sequence $(x_m) \in X^\infty$, $(d(x_m, x))$ is a real sequence in $\mathbb{R}_+$. Applying the definition above (with respect to the absolute value metric), we see that $d(x_m, x) \to 0$ if and only if for each $\varepsilon > 0$, there is a real number $M$ with $d(x_m, x) < \varepsilon$ for all $m \geq M$. Thus, the statement “$x_m \to x$” is the same thing as saying that “$d(x_m, x) \to 0$.” So, the apparently complicated matter of the convergence of a sequence in a metric space is always reduced to the apparently simple matter of the convergence of a particular real sequence. This allows us to extend many facts that we know about the convergence of real sequences to the context of sequences in an arbitrary metric space. For instance, evidently, $x_m \to x$ if and only if every subsequence of $(x_m)$ also converges to $x$. Similarly, but this is a bit more subtle, $x_m \to x$ if and only if every subsequence of $(x_m)$ has itself a subsequence that converges to $x$. 

30
5.2 Convergent Sequences in Product Metric Spaces

The observation we made in Example 5.3 extends to the case of all finite products of metric spaces. That is, a sequence in the product of finitely many metric spaces converges to a point in that space if and only if the corresponding sequence of the $i$th terms of that sequence converges to the $i$th term of the limit (in the $i$th metric space). As we show next, this is also true in the case of countably many products, which is the reason why some authors refer to the product metric as the metric of coordinatewise convergence.

Proposition 5.1. Let $((X_m, d_m))$ be a sequence of metric spaces. For any $x = x_1 \times x_2 \times \cdots$, and any sequence $(x_m)$ in the metric space $X^\infty(X_i, d_i)$, we have

$$x_m \to x \quad \text{if and only if} \quad x_{m,i} \to x_i \quad \text{for each} \quad i = 1, 2, \ldots,$$

where $x_{m,i}$ is the $i$th term of the sequence $x_m$ for each $i, m \in \mathbb{N}$.

Proof. To prove the “only if” part of the assertion, suppose $x_{m,j} \to x_j$ is false for some positive integer $j$. Then, we can find an $\varepsilon > 0$ and a strictly increasing sequence $(m_k)$ of positive integers such that the subsequence $(x_{m_k,j})$ stays out of the $\varepsilon$-ball around $x_j$ in $X_j$, that is, $d_j(x_{m_k,j}, x_j) \geq \varepsilon$ for each $k$.

But then

$$\rho(x_{m_k}, x) = \sum_{i=1}^{\infty} 2^{-i} \min \{1, d_i(x_{m_k,i}, x_i)\} \geq 2^{-j} \min \{1, d_j(x_{m_k,j}, x_j)\} \geq 2^{-j} \varepsilon$$

for each $k$. It follows that $(x_{m_k})$ is a subsequence of $(x_m)$ which does not converge to $x$, which means that $x_m \to x$ is false, as we sought.

To prove the “if” part of our assertion, assume that $x_{m,i} \to x_i$ for each $i$, and fix any $\varepsilon > 0$. We can obviously choose a positive integer $k \in \mathbb{N}$ large enough to guarantee that $2^{-(k+1)} + 2^{-(k+2)} + \cdots < \varepsilon/2$. (Why?) Since $d_i(x_{m,i}, x_i) \to 0$ for each $i$, we can also choose a $M > 0$ large enough so that

$$d_i(x_{m,i}, x_i) < \frac{\varepsilon}{2(2^{-1} + \cdots + 2^{-k})} \quad \text{for all} \quad m \geq M \text{ and } i = 1, \ldots, k.$$

But then, for each $m \geq M$,

$$\sum_{i=1}^{k} 2^{-i} \min \{1, d_i(x_{m,i}, x_i)\} < \frac{\varepsilon}{2} \quad \text{and} \quad \sum_{i=k+1}^{\infty} 2^{-i} \min \{1, d_i(x_{m,i}, x_i)\} < \frac{\varepsilon}{2},$$

and hence $\rho(x_m, x) < \varepsilon$. Since $\varepsilon > 0$ is arbitrary here, we may conclude that $x_m \to x$, as we sought.

Example 5.6. (The Hilbert Cube) Consider the set $[0, 1]^\infty$ of all real sequences all terms of which lie between 0 and 1. When we view $[0, 1]^\infty$ as a metric subspace of $\mathbb{R}^\infty$, that is, when we metrize this set by the product metric, the resulting metric space is called the Hilbert cube. In what follows we will spare the
notation \([0,1]^\infty\) for the Hilbert cube, that is, by default, we view \([0,1]^\infty\) as \(X^\infty([0,1],d_1)\).

The metric structure of the Hilbert cube is not the same as \(([0,1]^\infty,d_\infty)\).
For any \(x \in [0,1]^\infty\) and sequence \((x_m)\) in \([0,1]^\infty\), \(\|x_m - x\|_\infty \to 0\) implies \(x_{m,i} \to x_i\), and hence, by Proposition 5.1, we have \(\rho(x_m, x) \to 0\), where \(\rho\) is the product metric on \([0,1]^\infty\). The converse implication, however, does not hold.
To see this, consider the sequence \((x_m)\) of sequences where \(x_m = (\frac{1}{m}, \frac{1}{m-1}, ..., \frac{1}{2}, 1, 0, 0, ...)\), \(m = 2, 3, ...\)
and check that while \(x_{m,i} \to 0\) for each \(i\), we have \(\|x_m\|_\infty = 1\) for each \(m\). Thus \(\rho(x_m, (0,0,...)) \to 0\) but \((x_m)\) does not converge in \(([0,1]^\infty,d_\infty)\). We conclude:

A convergent sequence in \(([0,1]^\infty,d_\infty)\) is also convergent in the Hilbert cube \([0,1]^\infty\), but not conversely.

### 5.3 Sequential Characterization of Open/Closed Sets

There is a useful characterization of open sets in a metric space in terms of the convergent sequences in that space.

**Proposition 5.2.** A set \(O\) in a metric space \(X\) is open if, and only if, all but finitely many terms of any sequence that converges to a point in \(O\) belongs to \(O\).

**Proof.** Take any \(O \in \mathcal{O}_X\) and any sequence \((x_m)\) with \(x_m \to x\) for some \(x \in O\). Since \(O\) is open, there must be an \(\varepsilon > 0\) such that \(B(x, \varepsilon) \subseteq O\), and since \(x_m \to x\), there is a positive integer \(M\) such that \(x_m \in B(x, \varepsilon)\) for each \(m > M\). Thus, all but first \(M\) many terms of \((x_m)\) belong to \(O\). Conversely, suppose \(O\) is not open. Then, there is an \(x \in O\) such that \(B(x, \frac{1}{m})\) intersects \(X\backslash O\) for every positive integer \(m\). For each \(m\), pick any element \(x_m\) in the intersection of \(B(x, \frac{1}{m})\) and \(X\backslash O\), and notice that \((x_m)\) is a sequence in \(X\backslash O\) which converges to \(x\). Conclusion: If \(O\) is not closed, there is at least one sequence in \(X\backslash O\) that converges to a point in \(O\).

This result is easily reformulated to obtain a sequential characterization for closed sets.

**Proposition 5.3.** A set \(S\) in a metric space \(X\) is closed if, and only if, every sequence in \(S\) that converges in \(X\) converges to a point in \(S\).

**Proof.** Exercise.

To understand what this result says (or better, what it does not say), consider the metric space \((0,1)\) and ask yourself if \((0,1)\) is a closed subset of this space. A common mistake is to answer this question in the negative, and use Proposition 5.3 to suggest a proof. The fallacious argument goes as follows: “By Proposition 5.3, the interval \((0,1)\) cannot be closed, because the sequence \((1/m)\)
in $(0, 1)$ converges to 0, a point which is outside of $(0, 1)$.” The problem with this argument is that it works with a non-convergent sequence in $(0, 1)$. Indeed, $(1/m)$ does not converge anywhere in the space $(0, 1)$. After all, the only possible limit for this sequence is 0 (with respect to the absolute value metric), but 0 does not live in the mother space. In fact, any convergent sequence in $(0, 1)$ must converge in $(0, 1)$ (because of the funny structure of this metric space), and therefore we must conclude that $(0, 1)$ is closed as a subset of itself, which is, of course, a triviality. This observation points once again to the fact that the metric properties of sets (such as the convergence of sequences) depend crucially on the structure of the mother metric space under consideration.

**Example 5.7.** Combining what we have found in Example 5.6 and Proposition 5.3, we may conclude that any set which is closed in the Hilbert cube $[0, 1]^\infty$ is also closed in $([0, 1]^\infty, d_\infty)$, but not conversely. (Why?) That is, there are fewer closed (and hence open) sets in the Hilbert cube than in $([0, 1]^\infty, d_\infty)$.

It is also worth noting that we can characterize the closure, interior and boundary of a set in a metric space by using the convergent sequences in that space. We give such a characterization for the closure operator here, and leave the remaining two characterizations as exercises.

**Proposition 5.4.** Let $S$ be a subset of a metric space $X$. Then, $x \in \text{cl}(S)$ if, and only if, there is a sequence $(x_m)$ in $S$ that converges to $x$.

**Proof.** Suppose $x \in \text{cl}(S)$. Proposition 4.4 says that $B(x, \frac{1}{m})$ intersects $S$ for every $m \in \mathbb{N}$. Pick a point $x_m$ in this intersection for each $m$, and note that $d(x, x_m) < 1/m \downarrow 0$. Thus, $(x_m)$ is a sequence in $S$ that converges to $x$. Conversely, suppose there is an $(x_m) \in S^\infty$ with $x_m \to x$. Take any $\varepsilon > 0$. Choosing a positive integer $M$ large enough that $d(x, x_M) < \varepsilon$, we see that $B(x, \varepsilon) \cap S$ is nonempty because it contains $x_M$. In view of the arbitrariness of $\varepsilon$ and Proposition 4.4, therefore, $x \in \text{cl}(S)$.

### 5.4 Distance from a Set

**Definition.** Let $S$ be a nonempty subset of a metric space $X$. For any $x$ in $X$, we define **distance of $x$ from $S$** as

$$\text{dist}(x, S) := \inf_{\omega \in S} d(x, \omega).$$

(See Figure 1.9 for an illustration in $\mathbb{R}^2$.)

Obviously, $\text{dist}(x, S) = 0$ for every $x \in S$. But it is possible that a point that lies outside a given set be actually 0 distance away from that set. For instance, in $\mathbb{R}$, we have $\text{dist}(0, (0, 1)) = 0$. More generally:

**Proposition 5.5.** Let $X$ be a metric space and $S$ a subset of $X$. Then,

$$\text{cl}(S) = \{ x \in X : \text{dist}(x, S) = 0 \}.$$
**Proof.** If \( x \in \text{cl}(S) \), then, by Proposition 5.4, there is a sequence \((x_m)\) in \( S \) with \( x_m \to x \). But then \( d(x, x_m) \to 0 \), which means \( \inf_{\omega \in S} d(x, \omega) = 0 \), that is, \( \text{dist}(x, S) = 0 \). Conversely, if \( \text{dist}(x, S) = 0 \), then there must be a sequence \((x_m)\) in \( S \) with \( d(x, x_m) \to 0 \), that is, \( x_m \to x \). In view of Proposition 5.4, then, we have \( x \in \text{cl}(S) \).

**Exercises**

5.1. For each positive integer \( m \), consider the following real sequences:

\[
x_m := (0, \ldots, 0, m, m, \ldots) \quad \text{and} \quad y_m := (0, \ldots, 0, \frac{1}{m}, \frac{1}{m}, \ldots),
\]
where the only nonzero terms of these sequences are the first \( m - 1 \) terms. Do \((x_m)\) and \((y_m)\) converge in \( \mathbb{R}^1 \)? What if we consider \( \mathbb{R}^\infty \) as metrized instead by the metric introduced in Exercise 4.15?

5.2. Prove:
   a. The closure of \( c_{00} \) in \( \mathbb{R}^\infty \) is \( \mathbb{R}^\infty \).
   b. The closure of \( c_{00} \) is \( c_0 \) in \( (\mathbb{R}^\infty, D) \), where \( D \) is the metric introduced in Exercise 4.15.

5.3. For any positive integer \( m \), consider the continuous and bounded real maps \( f_m \) and \( g_m \) on \( \mathbb{R}_+ \) defined by

\[
f_m(x) := \frac{m x}{1 + x} \quad \text{and} \quad g_m(x) := \frac{m x}{1 + m^2 x}.
\]
When viewed as sequences in \( B(\mathbb{R}_+) \), determine if \((f_m)\) and \((g_m)\) converge.

5.4. Let \( X \) be a nonempty set, and make \( X^\infty \) a metric space as we described in Exercise 1.8. For each positive integer \( m \), take a sequence (of sequences) \((x_m)\) in \( X^\infty \). (We denote the \( k \)th term of the \( m \)th term of this sequence by \( x_{k,m} \).) Show that \((x_m)\) converges if and only if for every positive integer \( k \), there is an \( M(k) \in \mathbb{N} \) such that \( x_{m,i} = x_{M(k),i} \) for every \( i = 1, \ldots, k \) and \( m \geq M(k) \).

5.5. Give an example of a subset of \([0, 1]^\infty\) which is closed with respect to the sup-metric but not with respect to the product metric.
5.6. Prove: A sequence in a metric space converges iff every subsequence of that sequence converges.

5.7. Prove: A sequence in a metric space converges to a point iff every subsequence of that sequence itself has a subsequence that converges to that point.

5.8. Let $X$ be a metric space and $S$ a subset of $X$. We say that a point $x$ in $X$ is a limit point of $S$ if for every $\varepsilon > 0$, there is a point other than $x$ in $B(x, \varepsilon) \cap S$. Prove that $x$ is a limit point of $S$ iff there is a sequence of distinct points in $S$ that converges to $x$.

5.9. Show that every point of the Cantor set is a limit point of that set.

5.10. Let $(x_m)$ be a sequence in a metric space $X$. Prove:
   a. If $(x_m)$ is convergent, then $\{x_1, x_2, \ldots\}$ has exactly one limit point. (The converse of is false. For instance, $(1, 0, 2, 0, 3, 0, \ldots)$ has only one limit point.)
   b. Show that if $x$ is a limit point of $\{x_1, x_2, \ldots\}$, then there is a subsequence of $(x_m)$ that converges to $x$, but not conversely.

5.11. Let $X$ be a metric space, and take any $x \in X$. Prove:
   a. For any nonempty subsets $S$ and $T$ of $X$ with $S \subseteq T$, we have
      \[ \text{dist}(x, T) \leq \text{dist}(x, S) \leq \text{dist}(x, T) + \text{diam}(T). \]
   b. For any nonempty collection $\mathcal{S}$ of subsets of $X$, we have
      \[ \text{dist}(x, \bigcup \mathcal{S}) = \inf_{S \in \mathcal{S}} \text{dist}(x, S). \]

5.12. Show that $\inf(S) = \{x \in X : \text{dist}(x, X \setminus S) > 0\}$ for every subset $S$ of a metric space $X$.

5.13. $\partial S = \{x \in X : \text{dist}(x, S) = \text{dist}(x, X \setminus S)\}$ for every subset $S$ of a metric space $X$. True or False?

6 Equivalence of Metrics

When would endowing a given nonempty set $X$ with two different metrics $d$ and $D$ yield metric spaces that could reasonably be considered as “equivalent”? While this query is rather vague at present, we can still come up with a few benchmark responses to it. For instance, it makes perfect sense to view two metric spaces of the form $(X, d)$ and $(X, 2d)$ as “identical.” The second space simply equals a version of the former space in which the distances are measured in a different scale. Fine, how about $(X, d)$ and $(X, \sqrt{d})$? This comparison seems a bit more subtle. While $d$ and $\sqrt{d}$ are ordinally identical, one is not a simple rescaling of the other, so the metric spaces they induce may in principle look rather different from certain points of view. At the very least, it seems that the connection between $(X, d)$ and $(X, 2d)$ is tighter than that between $(X, d)$ and $(X, \sqrt{d})$, even though we would not expect the properties of the latter two spaces be vastly different from each other.
Let us be more precise now.

**Definition.** Let $d$ and $D$ be two metrics on a nonempty set $X$, and denote the collections of all open subsets of $X$ with respect to $d$ and $D$ as $\mathcal{O}(X,d)$ and $\mathcal{O}(X,D)$, respectively. We say that $d$ and $D$ (and/or $(X,d)$ and $(X,D)$) are **equivalent** if $\mathcal{O}(X,d) = \mathcal{O}(X,D)$, and that they are **strongly equivalent** if

$$\alpha d \leq D \leq \beta d$$

for some real numbers $\alpha, \beta > 0$.

As we proceed further in the course, it will become clear that the topology, that is, the collection of all open subsets, of a given metric space determines a good deal of the properties of this space. (In fact, as we shall see, in topology all that matters are these sets!) Consequently, if two metrics on a given nonempty set generate precisely the same collection of open sets, then the resulting metric spaces are bound to look “identical” from a variety of viewpoints. For instance, the collections of all closed subsets of two equivalent metric spaces are the same. Moreover, by Proposition 5.2, if a sequence in one metric space converges, then it also does so in any equivalent metric space.

**Example 6.1.** It is clear that if $(X,d)$ and $(X,D)$ are strongly equivalent metric spaces, then a subset $S$ of $X$ is bounded in $(X,d)$ iff it is bounded in $(X,D)$. By contrast, boundedness is not a property that is invariant under equivalence of metrics. Indeed, if $(X,d)$ is an unbounded metric space, then $(X,\min \{1,d\})$ is a bounded metric space, whereas $d$ and $\min \{1,d\}$ are equivalent. (Exercise: Prove this.) Thus, when $d$ is an unbounded function, $d$ and $\min \{1,d\}$ are equivalent metrics on $X$ that are not strongly equivalent.

**Example 6.2.** Two equivalent metric spaces $(X,d)$ and $(X,D)$ need not be strongly equivalent, even if $X$ is rendered bounded by both of these metrics. For instance, $([0,1],d_1)$ and $([0,1],\sqrt[3]{d_1})$ are equivalent (bounded) metric spaces. Yet, $d_1$ and $\sqrt[3]{d_1}$ are not strongly equivalent metrics on $[0,1]$. (Proof. There is no $\alpha > 0$ such that $\alpha^2 |x - y| \leq |x - y|^2$ for all $0 \leq x, y \leq 1$.)

**Example 6.3.** Let $n$ be a positive integer, and take any $p \in [1,\infty]$. Clearly, for any metric $d_p$ on $\mathbb{R}^n$ we have

$$d_\infty(x,y) \leq d_p(x,y) \leq nd_\infty(x,y) \quad \text{for all } x,y \in \mathbb{R}^n.$$  

Thus, $\mathbb{R}^{n,p}$ is strongly equivalent to $\mathbb{R}^{n,\infty}$ for any $p \in [1,\infty)$. Since “being strongly equivalent to” is an equivalence relation on the class of all distance functions on $\mathbb{R}^n$, we conclude: $d_p$ and $d_q$ are strongly equivalent metrics on $\mathbb{R}^n$ for any $p, q \in [1,\infty]$.\footnote{Due to this fact, it simply does not matter which $d_p$ metric is used to metrize $\mathbb{R}^n$, for the purposes of this text.}
Remark. While introducing the product of finitely many metric spaces in Section 3.1, we have noted that it would not matter for our purposes which of the formulas in (5), (6) (for any $p \geq 1$) and (7) we use for defining the product metric. We may now be more precise about this. What we mean is that any one of these formulas would turn in a distance function that is strongly equivalent to the metric given in (5). (The argument is identical to the one given in Example 6.3.) If we used any of these candidates as the definition of the product metric, the resulting space would have the same set of open sets, the same set of convergent sequences, the same set of bounded sets, the same interiors, closures, etc.

Remark. We can also use the notion of equivalence of metrics to demonstrate in what sense our metrization of countably infinitely many metric spaces is coherent with that of metrizing finitely many metric spaces. Let $((X_m, d_m))$ be a sequence of metric spaces, and fix an arbitrary point $z_i$ in $X_i$ for each $i$. For any positive integer $n$, let us agree to write $(x_1, ..., x_n, z)$ to mean the sequence $(x_1, ..., x_n, z_{n+1}, z_{n+2}, ...)$. Then, for any positive integer $n$, the space $X^n((X_i, d_i))$ is consistent with $X_n((X_i, d_i))$ in the following sense: If $X_n := X_1 \times \cdots \times X_n$ and $\rho_n$ stands for the real map on $X_n((X_i, d_i))$ defined by

$$\rho_n((x_1, ..., x_n), (y_1, ..., y_n)) := \rho((x_1, ..., x_n, z), (y_1, ..., y_n, z)),$$

then $\rho_n$ is a metric equivalent to the product metric on $X_n((X_i, d_i))$. We leave the (easy) proof of this as an exercise.

Exercises

6.1. Let $d$ and $D$ be two metrics on a nonempty set $X$. Prove: $d$ and $D$ are equivalent iff for every $x, x_1, x_2, ... \in X$, we have $d(x_m, x) \to 0$ iff $D(x_m, x) \to 0$. Conclude that $d$ and $\min\{1, d\}$ are equivalent metrics on $X$.

6.2. Let $X$ be a nonempty set. For any metrics $d$ and $D$ on $X$, let us write $d \preceq D$ if the following condition holds: For every $x \in X$ and $\varepsilon > 0$, there is a $\delta > 0$ such that $B_{X,d}(x, \delta) \subseteq B_{X,D}(x, \varepsilon)$. Prove: $d$ and $D$ are equivalent iff $d \preceq D$ and $D \preceq d$.

6.3. Let $d$ be a metric on a nonempty set $X$, and recall that $\frac{d}{1+d}$ is also a metric on $X$ (Exercise 1.11). Show that $d$ and $\frac{d}{1+d}$ are equivalent.

6.4. Let $X$ be a linear space, and take any two norms $\| \cdot \|_1$ and $\| \cdot \|_2$ on $X$. Show that $d_{\| \cdot \|_1}$ and $d_{\| \cdot \|_2}$ are equivalent iff they are strongly equivalent.

6.5. Let $n$ be a positive integer, and let $\| \cdot \|$ be any norm on $\mathbb{R}^n$. Prove that there is a real number $\theta > 0$ such that $\| \sum_{i=1}^n \lambda_i e_i \| \geq \theta \sum |\lambda_i|$ for every $\lambda_1, ..., \lambda_n \in \mathbb{R}$, where $e_1, ..., e_n$ are the standard basis vectors for $\mathbb{R}^n$. Then, use this fact to prove that the metrics induced by any two norms on $\mathbb{R}^n$ are strongly equivalent. (In Chapter 3, we will prove this result for an arbitrary finite-dimensional linear space.)
7 Separable Metric Spaces

7.1 Dense Sets

The following concept is fundamental.

Definition. Let $X$ be a metric space and $Z \subseteq X$. If $\text{cl}(Z) = X$, then $Z$ is said to be dense in $X$ (or a dense subset of $X$).

Thanks to Proposition 5.4, we see that a subset $Z$ of a metric space $X$ is dense in $X$ iff any point in $X$ can be approached by means of a sequence that is contained entirely in $Z$. Similarly, using Proposition 4.4, we see that $Z$ is dense in $X$ iff $B(x; \varepsilon) \cap Z \neq \emptyset$ for every $x \in X$ and $\varepsilon > 0$. (Evidently, this is the same thing as saying that $Z$ is dense in $X$ iff $Z$ intersects every nonempty open subset of $X$.) Another way of expressing this fact is:

$$Z \text{ is dense in } X \iff X = \bigcup_{z \in Z} B(z; \varepsilon) \text{ for every } \varepsilon > 0.$$  

(This follows readily from the fact that $z \in B(x; \varepsilon)$ iff $x \in B(z; \varepsilon)$.) We will use these alternate ways of thinking about dense sets routinely in what follows.

Example 7.1. Every metric space is dense in itself.

Example 7.2. The only dense subset of a discrete metric space is itself.

Example 7.3. As we have seen in Example 4.13, both $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$ are dense subsets of $\mathbb{R}$. Thus, for any real number there is a sequence of rational numbers that converges to that number, and the same holds for irrational numbers as well.

Example 7.4. For any positive integer $n$, a sequence in $\mathbb{R}^n$ converges to an $n$-vector $x$ iff the sequence of $i$th components of the sequence converges to $x_i$ for each $i$ (Example 5.3). As $\mathbb{Q}$ is dense in $\mathbb{R}$, therefore, using the sequential characterization of denseness, we find that $\mathbb{Q}^n$ is dense in $\mathbb{R}^n$.

The following generalizes the previous example.

Example 7.5. For any positive integer $n$, if $Y_i$ is a dense subset of a metric space $X_i$ for each $i = 1, \ldots, n$, then $Y_1 \times \cdots \times Y_n$ is a dense subset of $X_1 \times \cdots \times X_n$. (Why?)

7.2 Separable Metric Spaces

We now use the notion of denseness to introduce a very important class of metric spaces.
**Definition.** A metric space is said to be **separable** if it contains a countable dense set.

The alternate ways of looking at dense sets we have reviewed above provide a number of distinct, but equivalent, ways of thinking about separable metric spaces. In particular, it is evident that a metric space \( X \) is separable if there is a countable set in \( X \) which intersects every nonempty open subset of \( X \). Alternatively, \( X \) is separable if it contains a countable set \( Z \) such that every point of \( X \) can be approximated by points of \( Z \) to any desired degree of accuracy (that is, for any \( x \in X \), there is a sequence in \( Z \) that converges to \( x \)). Still another way of saying this is that \( X \) is separable if there is a sequence \((z_m)\) in \( X \) such that \( X = B(z_1, \varepsilon) \cup B(z_2, \varepsilon) \cup \cdots \) for every \( \varepsilon > 0 \).

Let us look at some examples.

**Example 7.6.** Every countable metric space is separable.

**Example 7.7.** As \( \mathbb{Q} \) is a countable dense set in \( \mathbb{R} \), we see that \( \mathbb{R} \) is separable. As the product of finitely many countable sets is countable, \( \mathbb{Q}^n \) is countable, and hence we may conclude that \( \mathbb{R}^n \) is separable as well. (\( \mathbb{R}^1 \) is separable too, but showing this requires a more subtle argument.) In fact, \( \mathbb{R}^{n; p} \) is separable for any \( p \in [1, \infty) \). (Why?)

**Example 7.8.** \( \ell_p \) is separable for any \( p \in [1, \infty) \). It may be tempting to suggest \( \ell_p \cap \mathbb{Q}^\infty \) as a candidate for a countable dense subset of \( \ell_p \). But this wouldn’t work, because, while this set is indeed dense in \( \ell_p \), it is not countable (since \( \mathbb{Q}^\infty \) is uncountable). Instead, we consider the set of those sequences in \( \mathbb{Q}^\infty \) all but finitely many components of which are zero. It is easy to check that this set, call it \( Y \), is countable. (Proof?) To verify that \( Y \) is dense, take any \( x \in \ell_p \) and fix any \( \varepsilon > 0 \). Since \( \sum_{i=1}^{\infty} |x_i|^p < \infty \), there must exist a positive integer \( M \) such that \( \sum_{i=M+1}^{\infty} |x_i|^p < \varepsilon^p/2 \). (Why?) Moreover, since \( \mathbb{Q} \) is dense in \( \mathbb{R} \), we can find a rational \( r_i \) such that \( |r_i - x_i|^p < \varepsilon^p/2M \) for each \( i = 1, \ldots, M \). But then

\[
d_p (x, (r_1, \ldots, r_M, 0, 0, \ldots)) = \left( \sum_{i=1}^{M} |r_i - x_i|^p + \sum_{i=M+1}^{\infty} |x_i|^p \right)^{1/p} < \left( \frac{\varepsilon^p}{2} + \frac{\varepsilon^p}{2M} \right)^{1/p} = \varepsilon.
\]

Thus any element of \( \ell_p \) can be approached by a sequence (of sequences) in \( Y \). (Corollary. \( c_{00} \) is dense in \( \ell_p \) for any \( p \in [1, \infty) \).)

**Identifying Non-Separable Metric Spaces**

There are many interesting non-separable metric spaces as well. The following observation is the main tool with which one identifies such spaces.
Proposition 7.1. A metric space \( X \) is not separable if there is an uncountable subset \( Y \) of \( X \) and an \( \varepsilon > 0 \) such that \( d(x, y) \geq \varepsilon \) for every distinct \( x, y \in Y \).

Proof. Suppose we can find such a \( Y \) and an \( \varepsilon > 0 \). Then, \( \{ B(x, \varepsilon/2) : x \in Y \} \) is an uncountable collection of open subsets of \( X \). Moreover, the members of this collection are pairwise disjoint. (Indeed, if \( z \) were to belong to \( B(x, \varepsilon/2) \) and \( B(y, \varepsilon/2) \) for distinct \( x, y \in Y \), we would get \( \varepsilon \leq d(x, y) \leq d(x, z) + d(z, y) < \varepsilon \), a contradiction.) But then no countable subset of \( X \) can possibly intersect every member of this collection. Conclusion: \( X \) is not separable.

For instance, a discrete metric space which contains uncountably many points is not separable. The following is a less trivial example.

Example 7.9. \( \ell_\infty \) is not separable. This follows readily from Proposition 7.1, because \( \{0, 1\}^\infty \) is an uncountable subset of \( \ell_\infty \), while \( d_\infty(x, y) = 1 \) for any distinct \( x \) and \( y \) in \( \{0, 1\}^\infty \). (Similarly, \( B(X) \) is not a separable metric space, unless \( X \) is a finite set.)

A Subspace of a Separable Metric Space is Separable

In practice, we often establish the separability of a metric space by showing that that space is in fact a metric subspace of another metric space which we already know to be separable. Indeed, separability of a metric space is inherited by any of its metric subspaces, as we prove next.

Proposition 7.2. Any metric subspace of a separable metric space is separable.

Proof. Let \( X \) be a separable metric space, and \( Y \) a metric subspace of \( X \). Take any countable dense subset \( Z \) of \( X \).\(^8\) For any positive integer \( m \), define

\[
A_m := \{ z \in Z : B(z, \frac{1}{m}) \cap Y \neq \emptyset \}.
\]

As \( Z \) is dense, we have \( X = \bigcup_{z \in Z} B(z, \frac{1}{m}) \), and therefore, \( Y \subseteq \bigcup_{z \in A_m} B(z, \frac{1}{m}) \), for every \( m \). Now, for any \( m \in \mathbb{N} \) and \( z \in A_m \), pick any \( w(z, m) \in B(z, \frac{1}{m}) \cap Y \), and put

\[
W := \bigcup_{m=1}^{\infty} \{ w(z, m) : z \in A_m \}.
\]

Since each \( A_m \) is countable, so is \( W \) (because the union of countably many countable sets is countable) and of course, we have \( W \subseteq Y \). To show that \( W \) is dense in \( Y \), take any \( y \in Y \) and \( \varepsilon > 0 \). Choose any \( m \geq 2/\varepsilon \), and note that there is a \( z \in A_m \) with \( y \in B(z, \frac{1}{m}) \). But then

\[
d(y, w(z, m)) \leq d(y, z) + d(z, w(z, m)) < \frac{1}{m} + \frac{1}{m} \leq \varepsilon.
\]

\(^8\)We wish to find a countable dense subset of \( Y \) by using \( Z \), but we cannot choose \( Z \cap Y \) for this purpose, for this set may even be empty. This is why the argument is a bit more subtle than what one might expect.
Conclusion: For any \( y \in Y \) and \( \varepsilon > 0 \), there is a \( w \in W \) such that \( y \in B(w, \varepsilon) \), that is, \( \text{cl}(W) = Y \).

Products of Separable Metric Spaces are Separable

Another method of generating new separable metric spaces is by taking the products of metric spaces which we already know to be separable. Indeed, as the product of finitely many countable sets is countable, it follows from Example 7.5 that the product of finitely many separable metric spaces is separable. The same is true for the products of countably infinitely many separable metric spaces.

**Proposition 7.3.** The product of countably many separable metric spaces is separable.

We sketch a proof for this in Exercise 7.9, but note that a more general result will be proved in Section 2.3 of Chapter 5.

**Exercises**

7.1. Let \( X \) and \( Y \) be two metric spaces, and let \( A \) and \( B \) be dense subsets of \( X \) and \( Y \), respectively. Show that \( A \times B \) is dense in \( X \times Y \). And then extend this observation inductively to prove the assertion we made in Example 7.5.

7.2. The classical Weierstrass Approximation Theorem says that the set of all polynomial functions on \([0, 1]\) is dense in \( C[0, 1] \). Argue that this result implies that \( C[0, 1] \) is separable.

7.3. Let \( X \) be a nonempty set. Show that \( B(X) \) is separable iff \( X \) is finite.

7.4. Recall Example 2.3 and prove that \( \ell^1, \ell^0 \) and \( c_0^0 \) are all separable metric subspaces of \( \ell^\infty \). (Hint: Consider the set of all eventually constant sequences of rational numbers.)

7.5. Let \( Y \) be a dense subset of a metric space \( X \). Now think of \( Y \) as a metric subspace of \( X \), and suppose \( Z \) is a dense subset of \( Y \). Show that \( Z \) must be dense in \( X \).

7.6. Let \( X \) be a metric space. Prove that \( X \) is separable iff there exists a countable collection \( \mathcal{O} \) of open subsets of \( X \) such that \( U = \bigcup \{ O \in \mathcal{O} : O \subseteq U \} \) for every open subset \( U \) of \( X \).

*7.7. Let \( X \) be an ultrametric space. Prove that \( X \) is separable iff the set of all open balls in \( X \) is countable.

7.8. Prove that “if” in Proposition 7.1 may be strengthened to “iff.”

7.9. (A Proof for Proposition 7.3) Let \((X_m, d_m)\) be a sequence of separable metric spaces, and for each \( i \), let \( Z_i \) be a countable dense set in \( X_i \). Fix any point \((x^*_1, x^*_2, \ldots)\) in \( X := X_1 \times X_2 \times \cdots \), and define

\[
Z := \bigcup_{k=1}^{\infty} \{(z_1, \ldots, z_k, x^*_{k+1}, x^*_{k+2}, \ldots) : z_i \in Z_i \text{ for each } i = 1, \ldots, k\}.
\]

Show that \( Z \) is a countable dense subset of \( X \).