CHARACTERIZATION OF CLOSED CONVEX SUBSETS OF $\mathbb{R}^n$

Chebyshev Sets

A subset $S$ of a metric space $X$ is said to be a **Chebyshev set** if, for every $x \in X$, there is a unique point in $S$ that is closest to $x$. Put differently, $S$ is Chebyshev iff

$$|\{\omega \in S : d(x, \omega) = d(x, S)\}| = 1 \quad \text{for every } x \in X.$$ 

For example, the one-dimensional unit sphere $S^1$ is not a Chebyshev subset of $\mathbb{R}^2$, for

$$\{\omega \in S^1 : d_2(0, \omega) = d_2(0, S^1)\} = S^1.$$ 

Similarly, $N_{1,\mathbb{R}^2}(0)$ is not Chebyshev. (If $x$ lies outside of $N_{1,\mathbb{R}^2}(0)$, then there is no point in $N_{1,\mathbb{R}^2}(0)$ that is closest to $x$.) On the other hand, we have seen in Example D.5 that, given any positive integer $n$, every nonempty closed and convex subset of $\mathbb{R}^n$ is, in fact, a Chebyshev subset of $\mathbb{R}^n$: It is remarkable that the converse of this also holds, that is, any Chebyshev set in $\mathbb{R}^n$ is a nonempty closed and convex subset of $\mathbb{R}^n$. It is this result that we wish to prove in this section.

Let’s first warm up with a few exercises.

**Exercise 1.** A subset $S$ of a metric space $X$ is said to be **proximinal** if, for every $x \in X$, there is at least one point in $S$ that is closest to $x$.

(a) Show that every proximinal set is closed;
(b) A closed set need not be proximinal (although this is the case in a Euclidean space). Let $X := \{(x_m) \in \ell^2 : x_m = 0 \text{ for all but finitely many } m\}$. Consider $X$ as a metric subspace of $\ell^2$, and show that $S := \{(x_m) \in X : \sum_{i=1}^{\infty} \frac{1}{i} x_i = 0\}$ is a closed subset of $X$ which is not proximinal.
(c) Every convex proximinal subset of $\mathbb{R}^n$ is Chebyshev.

**Exercise 2.** (Efimov-Stechkin) Let $S$ be a subset of a metric space $X$ and $x \in X$. A sequence $(y^n) \in S^\infty$ is said to be a **minimizing sequence for $x$ in $S$**, if $d(x, y^n) \to d(x, S)$. In turn, $S$ is said to be **approximately compact** if, for every $x \in X$, every minimizing sequence for $x$ in $S$ has a subsequence that converges in $S$.

(a) Show that every approximately compact set is proximinal.
(b) Show that $\{(x_m) \in \ell^2 : \sum_{i=1}^{\infty} |x_i| = 1\}$ is a proximinal subset of $\ell^2$ which is not approximately compact.

**Projection Operators**

First, let us note that the projection operator $p_S$ onto any Chebyshev subset $S$ of $\mathbb{R}^n$ is well-defined via the equation

$$d_2(x, p_S(x)) = d_2(x, S), \quad x \in X.$$ 

Remark G.4 shows that this operator is nonexpansive, provided that $S$ is convex. It is not readily clear if the same is true for any Chebyshev subset $S$ of $\mathbb{R}^n$, but we can at least say the following right away.
Lemma 1. The projection operator onto any Chebyshev subset of $\mathbb{R}^n$ is continuous.

Proof. Take any Chebyshev subset $S$ of $\mathbb{R}^n$, and let $(x^m)$ be a convergent sequence in $\mathbb{R}^n$ with $x := \lim x^m$. Note first that
\[
\begin{align*}
d_2(0, p_S(x^m)) &\leq d_2(0, x) + d_2(x, x^m) + d_2(x^m, S), \\
m = 1, 2, \ldots
\end{align*}
\]
It follows that $\{p_S(x^1), p_S(x^2), \ldots\}$ is a bounded set in $\mathbb{R}^n$. (Right?) Now, to derive a contradiction, suppose $\lim p_S(x^m) = p_S(x)$ is false. Then, since $\{p_S(x^1), p_S(x^2), \ldots\}$ is bounded, there must exist a convergent subsequence $(x^{m_k})$ of $(x^m)$ such that $y := \lim p_S(x^{m_k}) \neq p_S(x)$. But, by continuity of the maps $d_2(\cdot, S)$ and $d_2$,
\[
d_2(x, p_S(x)) = d_2(x, S) = \lim_{k \to \infty} d_2(x^{m_k}, S) = \lim_{k \to \infty} d_2(x^{m_k}, p_S(x^{m_k})) = d_2(x, y).
\]
Since $S$ is Chebyshev, this implies $p_S(x) = y$, a contradiction. ■

As another preliminary, we would like to make note of the following observation: If $S$ is a Chebyshev subset of $\mathbb{R}^n$ and $x$ is any point in $\mathbb{R}^n$, then the nearest point $p_S(x)$ in $S$ to $x$ is also the nearest point in $S$ to any point on the line segment between $p_S(x)$ and $x$. We prove this next.

Lemma 2. Let $S$ be a Chebyshev subset of $\mathbb{R}^n$ and $x \in \mathbb{R}^n$. Then,
\[
p_S(\lambda x + (1 - \lambda)p_S(x)) = p_S(x), \quad 0 \leq \lambda \leq 1.
\]

Proof. Suppose the claim is false, that is, there exists a $(\lambda, y) \in (0, 1) \times S$ with
\[
d_2(\lambda x + (1 - \lambda)p_S(x), y) < d_2(\lambda x + (1 - \lambda)p_S(x), p_S(x)).
\]
Then, by the triangle inequality,
\[
\begin{align*}
d_2(x, y) &< d_2(x, \lambda x + (1 - \lambda)p_S(x)) + d_2(\lambda x + (1 - \lambda)p_S(x), p_S(x)) \\
&= (1 - \lambda)d_2(x, p_S(x)) + \lambda d_2(x, p_S(x)) \\
&= d_2(x, p_S(x)) \\
&= d_2(x, S)
\end{align*}
\]
which is impossible. ■

\footnote{Wait, why? Because the closure of $\{p_S(x^1), p_S(x^2), \ldots\}$ is closed and bounded, and hence, it is compact by the Heine-Borel Theorem. Conclusion: Every subsequence of $(p_S(x^m))$ has a convergent subsequence. Therefore, if every convergent subsequence of $(p_S(x^m))$ converged to $p_S(x)$, it would follow that every subsequence of $(p_S(x^m))$ has a subsequence that converges to $p_S(x)$, which is just another way of saying $\lim p_S(x^m) = p_S(x)$.}
Motzkin’s Characterization of Convex Sets

We are now prepared to prove that every Chebyshev subset of $\mathbb{R}^n$ is convex. Originally proved by Theodore Motzkin in 1935, this is one of the gems of convex analysis.

**Motzkin’s Theorem.** For any positive integer $n$, a nonempty closed subset $S$ of $\mathbb{R}^n$ is Chebyshev if, and only if, it is convex.\(^2\)

Combining this result with Exercise 1 yields the characterization we promised above.

**Corollary.** A subset $S$ of $\mathbb{R}^n$ is Chebyshev if, and only if, it is nonempty, closed and convex.

Thanks to Motzkin’s Theorem, we can also strengthen Lemma 1 to the following:

**Corollary.** The projection operator onto any Chebyshev subset of $\mathbb{R}^n$ is nonexpansive.

*Proof.* Apply Motzkin’s Theorem and Remark G.4. ■

**Exercise 3.** Give an example of a metric $d$ such that a convex set in $(\mathbb{R}^2, d)$ is not Chebyshev.

**Exercise 4.** Give an example of a metric $d$ such that a non-convex set in $(\mathbb{R}^2, d)$ is Chebyshev.

The rest of this handout is devoted to the proof of Motzkin’s Theorem. Given Example D.5, all we need to do here is, then, to show that a Chebyshev set $S$ in $\mathbb{R}^n$ is convex. The crux of the argument is contained in the following fact: For any point $x$ in $\mathbb{R}^n$, the nearest point $p_S(x)$ in $S$ to $x$ is also the nearest point in $S$ to any point on the ray that begins at $p_S(x)$ and passes through $x$.

**Lemma 3.** Let $S$ be a Chebyshev subset of $\mathbb{R}^n$ and $x \in \mathbb{R}^n$. Then,

$$p_S(\lambda x + (1 - \lambda)p_S(x)) = p_S(x), \quad \lambda \geq 1. \quad (1)$$

Motzkin’s Theorem is easily proved by using Lemma 3. Let’s see this first. Take any Chebyshev subset $S$ of $\mathbb{R}^n$, and pick any two vectors $x$ and $y$ in $S$. For any given

\(^2\)A major open problem in approximation theory is if the role of $\mathbb{R}^n$ can be replaced with an arbitrary Hilbert space in this statement. (This is Klee’s problem.) While there are many partial answers to this query – for instance, it is known that an arbitrary pre-Hilbert space would not do – the status of the problem is open at present. (See Deutsch (2002) for more on this.)
0 < \lambda < 1$, we wish to show that $z := \lambda x + (1 - \lambda)y$ belongs to $S$. To this end, fix an arbitrary positive real number $\theta$, and notice that $(1 + \theta)z - \theta p_S(z) = z + \theta(z - p_S(z))$. Since $x \in S$, then, Lemma 3 maintains that

$$d_2(z + \theta(z - p_S(z)), p_S(z)) \leq d_2(z + \theta(z - p_S(z)), x)$$

that is,

$$(1 + \theta)^2 \sum_{i=1}^{n} (z_i - p_S(z)_i)^2 \leq \sum_{i=1}^{n} (z_i - x_i + \theta(z_i - p_S(z)_i))^2$$

where we denote that $i$th component of $p_S(z)$ as $p_S(z)_i$. Let’s open this up:

$$(1 + \theta)^2 \sum_{i=1}^{n} (z_i - p_S(z)_i)^2 \leq \sum_{i=1}^{n} (z_i - x_i)^2 + 2\theta \sum_{i=1}^{n} (z_i - x_i)(z_i - p_S(z)_i) + \theta^2 \sum_{i=1}^{n} (z_i - p_S(z)_i)^2$$

that is,

$$(1 + 2\theta) \sum_{i=1}^{n} (z_i - p_S(z)_i)^2 \leq \sum_{i=1}^{n} (z_i - x_i)^2 + 2 \sum_{i=1}^{n} (z_i - x_i)(z_i - p_S(z)_i).$$

Now divide both sides by $\theta$ and let $\theta \to \infty$ to get

$$(d_2(z - p_S(z), 0)) \leq (z - x)(z - p_S(z)).$$

We can obviously replace $x$ in this inequality by $y$ (or by any element of $S$ for that matter), so we also have

$$(d_2(z - p_S(z), 0)) \leq (z - y)(z - p_S(z)).$$

Aha! If we multiply the former inequality with $\lambda$ and the latter with $1 - \lambda$, and add them up, we get

$$(d_2(z - p_S(z), 0)) \leq (z - (\lambda x + (1 - \lambda)y))(z - p_S(z))$$

$$= (z - z)(z - p_S(z))$$

$$= 0.$$ 

But this means that $z = p_S(z)$, that is, $z \in S$, as we sought.

It remains to establish Lemma 3, which is a far more delicate matter. We shall attack the problem by first transforming it into a fixed point problem. To the best of my knowledge, this proof is due to Roger Webster.

**Proof of Lemma 3.** Let us suppose that the assertion of Lemma 3 is false. Then, there exists an $x'$ in $\mathbb{R}^n \setminus S$ such that

$$I := \{\lambda \geq 1 : p_S(\lambda x' + (1 - \lambda)p_S(x')) = p_S(x')\} \neq [1, \infty).$$

3To the best of my knowledge, this proof is due to Roger Webster.
But, by Lemma 2, $I$ has to be an interval with left-end point 1. (Yes?) By continuity of $p_S$ (Lemma 1), in turn, $I$ must contain its supremum. It follows that $I = [1, \alpha]$ for some real number $\alpha \geq 1$. Define $x := \alpha x' + (1 - \alpha)p_S(x')$. Then, $p_S(x) = p_S(x')$ and

$$p_S(\lambda x + (1 - \lambda)p_S(x)) \neq p_S(x) \quad \text{for any } \lambda > 1.$$  \hfill (2)

(This just shows that if Lemma 3 is false, then there is a point $x$ in $\mathbb{R}^n \setminus S$ such that, on the line that starts at $p_S(x)$ and passes through $x$, every point that is not on the line segment between $x$ and $p_S(x)$ has a projection onto $S$ different than $p_S(x)$.)

The rest is magic! Let $\beta := d_2(x, S)$, which is a positive number (as $S$ is closed (being proximinal) and $x$ lies outside $S$). Also define

$$K := \left\{ \omega \in \mathbb{R}^n : d_2(x, \omega) \leq \frac{\beta}{2} \right\},$$

which is a nonempty, closed, bounded and convex set in $\mathbb{R}^n$ that is disjoint from $S$. Now consider the function $\Phi : K \to \mathbb{R}^n$ defined by

$$\Phi(\omega) := x + \frac{\beta}{2} \frac{x - p_S(\omega)}{d_2(x, p_S(\omega))}.$$

Since $x \notin S$, this function is well-defined, and, by Lemma 1, it is continuous. (Yes?) Furthermore, for any $\omega \in K$, we have

$$d_2(x, \Phi(\omega)) = d_2 \left( \frac{\beta}{2} \frac{x - p_S(\omega)}{d_2(x, p_S(\omega))}, 0 \right) = \frac{\beta}{2d_2(x, p_S(\omega))} d_2(x, p_S(\omega)) = \frac{\beta}{2},$$

so that $\Phi(\omega) \in K$. Conclusion: $\Phi(K) \subseteq K$. It then follows from the Brouwer Fixed Point Theorem that $z = \Phi(z)$ for some $z \in K$. But this means that $z = x + \theta(x - p_S(z))$, where $\theta := \beta / 2d_2(x, p_S(z)) > 0$. Then

$$x = \frac{1}{1 + \theta} z + \frac{\theta}{1 + \theta} p_S(z),$$

that is, $x$ lies on the line segment that joins $z$ and $p_S(z)$. So, by Lemma 2, $p_S(z) = p_S(x)$, and hence $z = (1 + \theta)x + \theta p_S(x)$. Since $\theta > 0$, this contradicts (2). ■

**Exercise 5.** Let $S$ be a Chebyshev subset of $\mathbb{R}^n$ and $x \in \mathbb{R}^n \setminus S$. Prove that the following are equivalent without invoking Motzkin’s Theorem:

(i) $S$ is convex; (ii) $p_S$ is nonexpansive; and (iii) (1) holds for all $x \in \mathbb{R}^n \setminus S$.

*Exercise 6.* (Motzkin-Straus-Valentine) Let $S$ be a subset of $\mathbb{R}^n$ such that for every $x \in X$ there is a unique point in $S$ that is farthest away from $x$. Prove that $S$ is a singleton.
References

