1 Regular Spaces

The Hausdorff property is perhaps the most important “separation by open sets” property, but it is certainly not the only one worth studying. There are natural ways of strengthening this property, and a few of these pave the way for some of the deeper results of general topology. As we shall see, imposing such stronger separation properties avoids many sorts of pathologies, and leads us to topological spaces whose behavior are reminiscent of metric spaces in some important aspects.

Regularity

The first strengthening we will consider in this chapter is called regularity. While the Hausdorff property asks for separating two distinct points by two open sets, this property asks for the separation of a closed set from a point that lies in its outside in this manner.

**Definition.** Let \( X \) be a topological space. We say that \( X \) (or the topology of \( X \)) is **regular** if \( X \) is a \( T_1 \)-space, and for every closed set \( C \) in \( X \) and \( x \in X \setminus C \), there are disjoint open subsets \( O \) and \( U \) of \( X \) such that \( x \in O \) and \( C \subseteq U \). (In this case, we refer to \( X \) simply as a regular space.)

**Warning.** Some authors omit the \( T_1 \)-condition in the definition of regularity, and refer to what we call here a regular space as a “regular \( T_1 \)-space” or “\( T_3 \)-space.”

It is plain that regularity is a topological invariant. Moreover, since singletons are closed in any \( T_1 \)-space, every regular space is Hausdorff. As
such, regularity is a stronger “separation by open sets” property than being Hausdorff. Thus, we have the following hierarchy for topological spaces:

\[
\text{regularity} \implies \text{Hausdorff condition} \implies T_1\text{-condition}
\]

**Examples**

**Example 1.1.** Every discrete space is regular, while an indiscrete space is regular iff it contains a single point.

**Example 1.2.** Endowing an infinite set with the cofinite topology yields a \(T_1\)-space which is not regular.

**Example 1.3.** Every metric space \(X\) is regular. To see this, take any closed set \(C\) in \(X\) and \(x \in X\setminus C\). Then, there is an \(\varepsilon > 0\) such that \(B(x, \varepsilon) \cap C = \emptyset\). Put \(O := B(x, \varepsilon/2)\) and \(U := \bigcup_{y \in C} B(y, \varepsilon/2)\). Clearly, \(O\) and \(U\) are open sets in \(X\) with \(x \in O\) and \(C \subseteq U\). Moreover, these sets are disjoint. For, if \(z \in O \cap U\), then \(d(y, z) < \varepsilon/2\) for some \(y \in C\), but then, \(d(x, y) \leq d(x, z) + d(z, y) < \varepsilon\), which means \(y \in B(x, \varepsilon)\), contradicting \(B(x, \varepsilon)\) and \(C\) being disjoint.

**Example 1.4.** Every compact Hausdorff space is regular. (This follows from Lemma 1.4 of Chapter 7, but we will actually prove something more general than this shortly; see Corollary 1.2.)

**Example 1.5.** Let \(S\) stand for the collection \(\{(a, b) : a, b \in \mathbb{R}\} \cup \{\mathbb{Q}\}\), and make \(\mathbb{R}\) a topological space by using \(S\) as a subbasis. Let us denote the resulting topological space by \(X\). This space is Hausdorff, because its topology is finer than that of \(\mathbb{R}\). Now, \(\mathbb{R}\setminus \mathbb{Q}\) is a closed subset of \(X\). Take any rational number \(x\), and suppose there are disjoint open sets \(O\) and \(U\) in \(X\) such that \(x \in O\) and \(\mathbb{R}\setminus \mathbb{Q} \subseteq U\). Then, since \(O\) does not contain any irrational numbers, there must be an open interval \((a, b)\) such that \(x \in (a, b) \cap \mathbb{Q} \subseteq O\). But then if \(y\) is an irrational number in \((a, b)\), every open neighborhood of \(y\) in \(X\) – in particular, \(U\) – must intersect \((a, b) \cap \mathbb{Q}\), and hence \(O\), a contradiction. Conclusion: While Hausdorff, \(X\) is not regular.

**Warning.** It follows from this example that refining a regular topology may cause losing the regularity property (unlike the Hausdorff property).

**A Characterization of Regular Spaces**

The following characterization of regular spaces is used routinely.
Proposition 1.1. Let $X$ be a $T_1$-space. Then, the following are equivalent:

(a) $X$ is regular;

(b) For every $x$ in $X$ and every open neighborhood $O$ of $x$ in $X$, there is an open subset $U$ of $X$ such that

$$x \in U \subseteq \text{cl}(U) \subseteq O;$$

(c) For every closed set $C$ in $X$ and $x \in X \setminus C$, there is an open subset $V$ of $X$ such that

$$C \subseteq V \quad \text{and} \quad x \in X \setminus \text{cl}(V).$$

Proof. $(a) \Rightarrow (b)$. Take any $x \in X$ and $O \in \mathcal{O}_X(x)$. Then, $C := X \setminus O$ is a closed subset of $X$ with $x \in X \setminus C$. By (a), there are disjoint open subsets $U$ and $V$ of $X$ such that $x \in U$ and $C \subseteq V$. Then $\text{cl}(U) \subseteq X \setminus V \subseteq X \setminus C = O$, so $U$ satisfies the requirements of (b).

$(b) \Rightarrow (c)$. Take any closed subset $C$ of $X$ and $x \in X \setminus C$. Put $O := X \setminus C$, and apply (b) to find a $U \in \mathcal{O}_X(x)$ such that $\text{cl}(U) \subseteq O$. Put $V := X \setminus \text{cl}(U)$, which is an open subset of $X$. Then, $V \subseteq X \setminus U$, so $\text{cl}(V) \subseteq X \setminus U$, so, in particular, $x \in X \setminus \text{cl}(V)$. Besides, as $\text{cl}(U) \subseteq X \setminus C$, we have $C = X \setminus (X \setminus C) \subseteq X \setminus \text{cl}(U) = V$.

$(c) \Rightarrow (a)$. Take any closed set $C$ in $X$ and $x \in X \setminus C$. By (c), there is an open subset $U$ of $X$ such that $C \subseteq U$ and $x \in X \setminus \text{cl}(U)$. Put $O := X \setminus \text{cl}(U)$, which is an open subset of $X$. Then, $O \cap U = \emptyset$ and $x \in O$. In view of the arbitrary choice of $x$ and $C$, and because $X$ is assumed to be Hausdorff, therefore, we may conclude that $X$ is regular.

Corollary 1.2. Every locally compact Hausdorff space is regular.

Proof. Apply Propositions 4.1 of Chapter 7 and Proposition 1.1.

Inheritance Properties of Regular Spaces

There is not a whole lot of interesting things to say about regular spaces at this level of generality. (We really view this property as a stepping stone to the attractions coming later in the chapter.) But it is worth noting that regularity is a property that behaves quite well when we form subspaces and products. Indeed, it is readily checked that regularity is hereditary. Moreover, the product of any collection of regular spaces is regular:

Proposition 1.3. Let $I$ be a nonempty set, $X_i$ a topological space for each $i \in I$, and $X$ the topological product of $\{X_i : i \in I\}$. Then, $X$ is regular if, and only if, $X_i$ is regular for each $i \in I$. 

3
Proof. Suppose $X$ is regular, and take any $i \in I$. For every $j \in I \setminus \{i\}$, we pick any $x_j \in X_j$, and note that $X_i$ is homeomorphic to the topological product of $\{X_i\} \cup \{\{x_j\} : j \in I \setminus \{i\}\}$. The latter space is regular, because it is a subspace of $X$, and regularity is hereditary. It follows that $X_i$ is regular (because regularity is a topological invariant). Conversely, suppose that $X_i$ is regular for each $i \in I$. Then, $X$ is a $T_1$-space (Proposition 1.3 of Chapter 5). To prove the regularity of $X$, we will use Proposition 1.1. So, take any $x \in X$ and any $O \in \mathcal{O}_X(x)$. Then, there is a finite subset $J$ of $I$, and for each $i \in I$, an open subset $O_i$ of $X_i$, such that $x \in \bigcap_{i \in J} O_i \subseteq O$ and $O_i = X_i$ for all $i \in I \setminus J$. Since $X_j$ is regular, Proposition 1.1 says that there is an open subset $U_j$ of $X_j$ such that $x(j) \in U_j \subseteq \text{cl}(U_j) \subseteq O_j$, for every $j \in J$. Setting $U_i := X_i$ for every $i \in I \setminus J$, therefore, we have

$$x \in \bigcap_{i \in I} U_i \subseteq \text{cl} \left( \bigcap_{i \in I} U_i \right) \subseteq X \text{cl}(U_i) \subseteq O.$$ 

In view of the arbitrary choice of $x$ and $O$, and Proposition 1.1, we may thus conclude $X$ is regular.

Exercises

1.1. Prove that regularity is a topological invariant and is hereditary.

1.2. Show that the Smirnov topology is Hausdorff but not regular.

1.3. Let $X$ be a topological space, and $Y$ and $Z$ disjoint subspaces of $X$ such that $X = Y \sqcup Z$. Show that if $Y$ and $Z$ are regular, then so is $X$.

1.4. Let $X$ be a regular space, and $C$ a closed subset of $X$. Prove: $C = \bigcap\{O \in \mathcal{O}_X : C \subseteq O\}$.

1.5. Let $X$ be a regular space, and $C$ a closed subset of $X$. Show that $X/C$ is Hausdorff.

1.6. Put $X := [0, 1] \times \{0, 1\}$, and let $\sim$ be the smallest equivalence relation on $X$ such that $(t, 0) \sim (t, 1)$ for all $t \in [0, 1)$. Show that $\pi_\sim$ is an open map, and $X/\sim$ is a $T_1$-space, but $X/\sim$ is not regular. (Thus, regularity is not preserved under open and continuous maps. The same is true under closed and continuous maps as well, but this is harder to demonstrate.)

1.7. Let $X$ be a regular space and $\sim$ an equivalence relation on $X$. Show that if $\pi_\sim$ is a closed map, then $\sim$ is a closed subset of $X \times X$. Also give an example to show that the converse of this is false.
2 Normality

2.1 Normal Spaces

First Impressions

Regularity of a topological space allows us to separate a closed subset of that space from a point that lies in its complement by means of two disjoint open sets. A natural, and far more rewarding, generalization of this property (at least for \( T_1 \)-spaces) is the ability to separate any two disjoint closed sets in this manner. Almost all spaces that are encountered in practice satisfy this stronger separation property, and are thus given a special name.

Definition. Let \( X \) be a topological space. We say that \( X \) (or the topology of \( X \)) is normal if \( X \) is a \( T_1 \)-space and for every two disjoint closed sets \( A \) and \( B \) in \( X \), there are disjoint open sets \( O \) and \( U \) in \( X \) with \( A \subseteq O \) and \( B \subseteq U \). (In this case, we refer to \( X \) simply as a normal space.)

Warning. Some authors omit the \( T_1 \)-condition in the definition of normality. (According to that definition, for instance, the Sierpiński space would be declared normal. According to our definition, however, the Sierpiński space is not normal, because it is not a \( T_1 \)-space.) Also note that normal spaces are sometimes referred to as “\( T_4 \)-spaces” in the literature.

Once again, normality is a topological invariant. Moreover, since singletons are closed in any \( T_1 \)-space, every normal space is regular. Thus, we have the following hierarchy for topological spaces:

\[
\text{normality} \implies \text{regularity} \implies \text{Hausdorff condition} \implies T_1\text{-condition}
\]

Examples

**Example 2.1.** Normality and regularity are the same properties for discrete and indiscrete spaces. That is, every discrete space is normal, while an indiscrete space is normal iff it contains a single point.

**Example 2.2.** *The Sorgenfrey line is normal.* Since the topology of the Sorgenfrey line is finer than that of \( \mathbb{R} \), it is obviously a Hausdorff (hence \( T_1 \)-) space. Now take any two disjoint closed sets \( A \) and \( B \) in the Sorgenfrey line. Since \( \mathbb{R} \setminus B \) is open in this space, for each \( a \in A \) there must be a real number \( t_a > a \) such that \( [a, t_a) \) is contained in \( \mathbb{R} \setminus B \). Similarly, for each \( b \in B \) there is a real number \( s_b > b \) such that \( [b, s_b) \cap A = \emptyset \). Consequently,
$O := \bigcup_{a \in A} [a, t_a)$ and $U := \bigcup_{b \in B} [b, s_b)$ are two disjoint open sets in the Sorgenfrey line such that $A \subseteq O$ and $B \subseteq U$.

**Example 2.3.** Let $(X, \succ)$ be a loset. Then, $X$ is a normal space relative to the $\succ$-order topology. (Exercise!)

**Example 2.4.** Every metric space is normal. A proof for this was already sketched in Exercise 4.20 of Chapter 1.

**Example 2.5.** Every compact Hausdorff space $X$ is normal. This follows from Proposition 1.6 of Chapter 7, and the fact that Hausdorff property ensures that $X$ is a $T_1$-space.

We have seen in Corollary 1.2 that every locally compact Hausdorff space is regular. Given the previous example, it is thus natural to wonder if every such space is normal as well. However, we will see below that this is not true in general.

**A Characterization of Normal Spaces**

The following characterization of normal spaces is used routinely.

**Proposition 2.1.** Let $X$ be a $T_1$-space. Then, the following are equivalent:

(a) $X$ is normal;

(b) For every closed subset $C$ of $X$ and every open subset $O$ of $X$ with $C \subseteq O$, there is an open subset $U$ of $X$ such that

$$C \subseteq U \subseteq \text{cl}(U) \subseteq O;$$

(c) For every disjoint closed sets $A$ and $B$ in $X$, there is an open subset $V$ of $X$ such that

$$B \subseteq V \quad \text{and} \quad A \cap \text{cl}(V) = \emptyset.$$

The proof of this result is quite similar to that of Proposition 1.1; we leave it as an exercise.

**Inheritance Properties of Normal Spaces**

The property of normality does not behave particularly well in terms of forming subspaces and products. For one thing, normality is not a hereditary property, in general. (Proving this by means of a counter-example is not
easy, however.) On the other hand, all goes well with respect to closed subspaces. Indeed, using the fact that a closed subset of a closed subspace of a topological space is closed in that space, we can easily prove that every closed subspace of a normal topological space is normal.

**Remark.** With some serious effort, one can show that $\mathbb{R}^I$ is not normal for any uncountable (index) set $I$. (We omit the proof of this fact here.) As $(0,1) \cong \mathbb{R}$, we have $(0,1)^I \cong \mathbb{R}^I$, and hence, $(0,1)^I$ is not normal. On the other hand, $[0,1]^I$ is normal, because this is a compact Hausdorff space (by the Tychonoff Theorem). Thus: $(0,1)^I$ is a non-normal subspace of the normal space $[0,1]^I$.

Unlike the case with regularity, the situation with products of normal spaces is fairly troublesome. It turns out that the product of even two normal spaces need not be normal. We will use the following criterion of non-normality to establish this fact:

**Jones' Lemma.** Let $X$ be a normal space. If $Z$ is a dense subset of $X$, and $Y$ is a closed subset of $X$ which is a discrete space (relative to the subspace topology), then $2^Z \not\leq_{\text{card}} 2^Y$.

**Proof.** Note first that every subset of $Y$ is closed in $Y$ (because $Y$ is discrete), and hence in $X$ (because $Y$ is closed in $X$). It follows from normality of $X$ that for every subset $A$ of $Y$, there exist disjoint open subsets $O_A$ and $U_A$ of $X$ such that $A \subseteq O_A$ and $Y \setminus A \subseteq U_A$. We may then define the map $f : 2^Y \to 2^Z$ by $f(A) := O_A \cap Z$. We claim that this map is injective. To see this, take any distinct subsets $A$ and $B$ of $Y$. Then, either $B$ does not contain $A$, or conversely. Relabelling if necessary, we may assume that $B$ does not contain $A$. Then, $O_A \cap U_B$ is a nonempty open subset of $X$, so, as $Z$ is dense in $X$, we have $O_A \cap U_B \cap Z \neq \emptyset$. Since $O_B \cap Z$ and $U_B \cap Z$ are disjoint, we thus find that $O_A \cap Z$ and $O_B \cap Z$ are distinct. We thus conclude that $f$ is injective. Conclusion: $2^Z \not\leq_{\text{card}} 2^Y$.

**Corollary 2.2.** Let $X$ be a separable topological space. If there is an uncountable closed subset of $X$ which is a discrete space (relative to the subspace topology), then $X$ cannot be normal.

**Proof.** Let $Z$ be a countable dense set in $X$, and suppose $Y$ is an uncountable closed subset of $X$ which is a discrete space (as a subspace of $X$). Then, $Y \not\leq_{\text{card}} Z$, so we have $2^Y \not\leq_{\text{card}} 2^Z$ (see Appendix). Thus, by Jones’ Lemma, $X$ cannot be normal.
Example 2.6. *(The Sorgenfrey Plane)* Let us denote the Sorgenfrey line by $X$. We have seen in Example 2.2 that this space is normal. By contrast, $X \times X$ – this space is called the Sorgenfrey plane – is not normal. Indeed, the Sorgenfrey plane is separable (since, for instance, $\mathbb{Q}^2$ is dense in $X \times X$). On the other hand, $Y := \{x \in \mathbb{R}^2 : x_1 = -x_2\}$ is an uncountable subset of $X \times X$ which is readily checked to be closed. Besides, $Y$ is a discrete subspace of $X \times X$ (because $Y \cap ([a, \infty) \times [-a, \infty)) = \{(a, -a)\}$ for every real number $a$). By Corollary 2.2, therefore, the Sorgenfrey plane is not normal.

Remark. This example also shows that a regular space need not be normal. Indeed, as the Sorgenfrey line is regular, Sorgenfrey plane is regular as well (Proposition 1.3), but we have just found that it is not normal.

A Locally Compact Non-normal Space

Example 2.7. *(The Rational Sequence Topology)* For every irrational number $x$, let us associate a sequence $(q_m(x))$ of distinct rational numbers such that $q_m(x) \to x$. Then, let $\mathcal{B}$ stand for the collection that consists of all finite subsets of $\mathbb{Q}$ as well as all sets of the form $\{q_m(x), q_{m+1}(x), \ldots, x\}$, where $m$ is a positive integer and $x$ an irrational number. It is an easy exercise to check that $\mathcal{B}$ is a basis for a topology on $\mathbb{R}$. Let us now endow $\mathbb{R}$ with the topology generated by $\mathcal{B}$ – this topology is known as the rational sequence topology – and denote the resulting (Hausdorff) topological space by $X$.

Notice that if $\mathcal{O}$ is an open cover of a set like $\{q_m(x), q_{m+1}(x), \ldots, x\}$ in $X$, where $m \in \mathbb{N}$ and $x \in \mathbb{R}\setminus\mathbb{Q}$, then there must exist a $k \geq m$ such that one of the members of $\mathcal{O}$ contains $\{q_k(x), q_{k+1}(x), \ldots, x\}$, which implies that a finite cover of the first set can be extracted from $\mathcal{O}$. As all other basis elements are finite sets, therefore, we find that the topology of $X$ is generated by a basis that consists of compact sets. Thus: $X$ is locally compact. On the other hand, $X$ is separable (as $\mathbb{Q}$ is dense in $X$), and $\mathbb{R}\setminus\mathbb{Q}$ is an uncountable discrete subspace of $X$ (because, for any irrational number $x$, we have $\{q_k(x), q_{k+1}(x), \ldots, x\} \cap \mathbb{R}\setminus\mathbb{Q} = \{x\}$). Finally, notice that $\mathbb{Q}$ is open in $X$, being the union of open subsets of $X$, so $\mathbb{R}\setminus\mathbb{Q}$ is closed in $X$. Therefore, by Corollary 2.2, $X$ is not normal.

Since every locally compact Hausdorff space is regular, the example above provides another illustration of a regular space which is not normal.

Regularity plus Second Countability imply Normality
While normality is in general stronger than regularity, these properties coincide for second-countable spaces. We prove this result, which is sometimes referred to as Tychonoff’s Lemma, next.

**Proposition 2.3.** Let $X$ be a second-countable topological space. Then, $X$ is normal if, and only if, it is regular.

**Proof.** Assume that $X$ is regular, and let $\mathcal{B}$ be a countable basis for the topology of $X$. To prove that $X$ is normal, take any disjoint closed subsets $A$ and $B$ of $X$. For each $x$ in $A$, $X \setminus B$ is an open neighborhood of $x$ in $X$, so by regularity of $X$ (Proposition 1.1), there is an open neighborhood $O_x$ of $x$ in $X$ such that $\text{cl}(O_x) \cap B = \emptyset$. Clearly, it is without loss of generality to assume that each $O_x$ belongs to $\mathcal{B}$. Since $\mathcal{B}$ is countable, therefore, we conclude that there is a countable collection $O$ of open sets in $X$ which covers $A$ and satisfies $\text{cl}(O) \cap B = \emptyset$ for each $O \in \mathcal{O}$. Analogously, we find a countable collection $U$ of open sets in $X$ which covers $B$ and satisfies $\text{cl}(U) \cap A = \emptyset$ for each $U \in \mathcal{U}$. Let us enumerate $\mathcal{O}$ and $\mathcal{U}$ as $\{O_1, O_2, \ldots\}$ and $\{U_1, U_2, \ldots\}$, respectively, and define

\[
O'_m := O_m \setminus (\text{cl}(O_1) \cup \cdots \cup \text{cl}(U_m)) \quad \text{and} \quad U'_m := U_m \setminus (\text{cl}(O_1) \cup \cdots \cup \text{cl}(O_m))
\]

for each positive integer $m$. Then, $O := \bigcup^\infty O'_i$ and $U := \bigcup^\infty U'_i$ are disjoint open sets in $X$ such that $A \subseteq O$ and $B \subseteq U$.

### 2.2 Urysohn’s Lemma

We have so far looked at a few examples and basic properties of normal spaces, but did not really put an effort to explain why such spaces are of interest. (This is partly responsible for the exposition of the present chapter being somewhat on the boring side so far.) But things are about to change dramatically on that score.

Let us pose the following problem: Is there always a non-constant continuous real map on a topological space? The answer to this question at this level of generality is readily seen to be no. Indeed, on any indiscrete space, a real map is continuous if and only if it is constant. But, as indiscrete spaces are pathological, one is naturally tempted to dismiss this observation as an anomaly. Yet, the issue gets more pressing if we ask the underlying topological space to be, say, Hausdorff. Indeed, it turns out that the Hausdorff property is not strong enough to turn in a positive answer to our question. In fact, it is known that there are even (nonsingleton) regular spaces on which every
continuous real map is constant.\footnote{There is a famous example, due to Erwin Hewitt, that shows this, but this example (and all derivatives of it that I am aware of) are quite complicated.}

On the other hand, there are plenty of examples of spaces for which we get a positive answer. For instance, in the case of any metric space $X$, all goes well. Indeed, for any two nonempty disjoint closed subsets $A$ and $B$ of $X$, the map $f : X \to [0, 1]$ defined by

$$f(x) := \frac{\text{dist}(x, A)}{\text{dist}(x, A) + \text{dist}(x, B)},$$

is continuous, and satisfies $f|_A = 0$ and $f|_B = 1$. (You were asked to prove these claims in Exercise 1.13 of Chapter 2.)

So, regularity of a space $X$ is not enough for ensuring the nontriviality of $C(X)$, but its metrizability is. This is not bad, but the gap between regularity and metrizability is quite large, so this finding is far from being tight. It turns out that normality, which sits between these two properties, settles the problem beautifully. In fact, not only that almost all spaces of practical interest are normal, but $C(X)$ is quite rich for such spaces. After all, one of the most prominent results of general topology says that what we have just found in the context of metric spaces is valid for normal spaces as well. As a matter of fact, this property characterizes normality.

**Urysohn’s Lemma.** Let $X$ be a normal space, and $A$ and $B$ two disjoint closed subsets of $X$. Then, there is a continuous map $f : X \to [0, 1]$ such that $f|_A = 0$ and $f|_B = 1$.

**Remark.** The choice of the numbers 0 and 1 in this definition is only for concreteness; all that matters is that these numbers are distinct. (Indeed, if $f : X \to [0, 1]$ is a continuous map such that $f|_A = 0$ and $f|_B = 1$, and $a$ and $b$ are any two distinct real numbers, then $g := (b-a)f + a$ is a continuous real map on $X$ such that $g|_A = a$ and $g|_B = b$.)

**Historical Note.** Pavel Urysohn (1898-1924) was an extraordinarily talented mathematician. Along with Pavel Alexandroff, he is considered as one of the two founders of the Russian school of topology. His impact on the development of general topology has been remarkable. In fact, Urysohn’s Lemma is sometimes referred to as “the first nontrivial fact of point set topology.”

Urysohn published his first paper when he was 17 (on a physics topic), and completed his Ph.D., under the supervision of Nikolai Lusin, in 1921. His name is attached to two phenomenal results in topology, one is the existence lemma that we have just stated, and the other is a metrization theorem which we will study.
a bit later. Urysohn also made lasting contributions to what is called dimension theory, a major branch of general topology. All this, in his extremely short career. Urysohn died when he was only 26 – he apparently drowned while swimming off a coast of Brittany, France.

Proof of Urysohn’s Lemma

As a normal space need not be metrizable – think of the Sorgenfrey line – we cannot adopt the method of proof we used above in the case of metric spaces. Indeed, the proof is more complicated in the general case, but it is still based on a particular method. As this method is used in other contexts as well, it is surely worth your while to learn it well.

Before we move on to our main course, let us recall the notion of a dyadic rational number. We put

\[ D_0 := \{0, 1\}, \quad D_1 := \left\{ \frac{1}{2} \right\}, \quad D_2 := \left\{ \frac{1}{4}, \frac{3}{4} \right\}, \quad D_3 := \left\{ \frac{1}{5}, \frac{3}{5}, \frac{5}{5}, \frac{7}{5} \right\}, \]

and so on. (That is, \( D_0 := \{0, 1\}, \ D_1 := \{\frac{1}{2}\} \) and \( D_n := \{k2^{-n} : k = 1, 3, ..., 2^n - 1\} \) for every \( n = 2, 3, ... \).) Next, we define \( D := D_0 \cup D_1 \cup \cdots \). Any one member of this set is called a dyadic rational number in \([0, 1]\). Such numbers serve as useful tools for carrying out various types of constructions in mathematical analysis, and we will see a brilliant illustration of this shortly.

We would like to make two preliminary observations about the dyadic rational numbers in \([0, 1]\). First, take any integer \( n \geq 2 \), and any \( k \in \{1, 3, ..., 2^n - 1\} \). (Thus, \( k2^{-n} \in D_n \)) Since \( k \) is odd, \( k - 1 = 2a \) for some \( a \in \{0, 1, ..., 2^{n-1} - 1\} \). If \( a \) is odd, then \( (k - 1)2^{-n} = a2^{-n+1} \in D_{n-1} \). Otherwise, we may repeat this reasoning to eventually find that \( (k - 1)2^{-n} \in D_m \) for some \( m \in \{0, ..., n - 2\} \). Since \( k \) is odd, exactly the same argument can be carried out to find that \( (k + 1)2^{-n} \in D_m \) for some \( m \in \{0, ..., n - 1\} \). Conclusion: For any integer \( n \geq 2 \) and \( k \in \{1, 3, ..., 2^n - 1\} \),

\[ \{(k - 1)2^{-n}, (k + 1)2^{-n}\} \subseteq \bigcup_{m=0}^{n-1} D_m. \]  \( \text{(1)} \)

In other words, the two numbers adjacent to \( k2^{-n} \) in \( D_n \) belong to \( D_0 \cup \cdots \cup D_{n-1} \). This highlights the recursive hierarchy that the sets \( D_0, D_1, \ldots \) possess.

Second, notice that if \( a \) and \( b \) are two real numbers in \([0, 1]\) with \( b > a \), then there is a positive integer \( n \) with \( b - a > 2^{-n} \). Then, for any integer \( k \) with \( k > 2^n a \geq k - 1 \), we have \( b > a + 2^{-n} \geq (k - 1)2^{-n} + 2^{-n} = k2^{-n} > a \).
that is, \(k2^{-n}\) is a dyadic rational number in \([0, 1]\) that belongs to the interval \((a, b)\). Thus: \(D\) is dense in \([0, 1]\).

**Proof of Urysohn’s Lemma.** Let us put \(O_0 := A\). In the first step of the proof, we claim that one can assign an open subset \(O_d\) of \(X\) to each \(d \in D\) such that

\[
\text{cl}(O_t) \subseteq O_s
\]

for every \(s, t \in D\) with \(s > t\). Recalling that \(D\) equals the union of the sets \(D_0 \cup \cdots \cup D_n\), where \(n \in \mathbb{Z}_+\), we will prove our claim by defining our \(O_d\)s by induction on \(n\). We have already defined \(O_0\); we now set \(O_1 := X \setminus B\). (This defines \(O_d\) for each \(d \in D_0\).) Since \(O_0\) is a closed set that is contained in the open set \(O_1\) (because \(A\) and \(B\) are disjoint), and \(X\) is normal, we may use Proposition 2.1 to find an open subset \(O_{1/2}\) of \(X\) such that

\[
O_0 \subseteq O_{1/2} \subseteq \text{cl}(O_{1/2}) \subseteq O_1.
\]

(This defines \(O_d\) for each \(d \in D_0 \cup D_1\).) Now fix any integer \(n \geq 2\), and assume (as our induction hypothesis) that we have defined \(O_d\) for every \(d \in D_0 \cup \cdots \cup D_{n-1}\). Take any \(t \in D_n\) so that \(t = k2^{-n}\) for some \(k \in \{1, 3, ..., 2^n - 1\}\). In view of (1), we know that \(O_{(k-1)2^{-n}}\) and \(O_{(k+1)2^{-n}}\) have already been defined in such a way that

\[
\text{cl}(O_{(k-1)2^{-n}}) \subseteq O_{(k+1)2^{-n}}.
\]

By Proposition 2.1, therefore, there is an open subset \(O_t\) of \(X\) such that

\[
\text{cl}(O_{(k-1)2^{-n}}) \subseteq O_t \subseteq \text{cl}(O_t) \subseteq O_{(k+1)2^{-n}}.
\]

Given the arbitrary choice of \(t \in D_n\), therefore, we conclude that \(O_d\) is defined for each \(d \in D_0 \cup \cdots \cup D_n\) such that \(\text{cl}(O_t) \subseteq O_s\) for every \(s, t \in D_0 \cup \cdots \cup D_n\) with \(s > t\). By the principle of mathematical induction, our claim is proved.

We now move to the second step of the proof. Define \(f : X \to [0, 1]\) by

\[
f(x) := \begin{cases} 
\inf \{t \in D : x \in O_t\}, & \text{if } x \in X \setminus B \\
1, & \text{otherwise.}
\end{cases}
\]

Then, as \(A = O_0\), we have \(f|_A = 0\), and obviously, \(f|_B = 1\). We will complete our proof by showing that \(f\) is continuous. Since the topology of \([0, 1]\) is generated by using \(\{(a, 1] : 0 \leq a < 1\} \cup \{(0, b) : 0 < b \leq 1\}\) as a subbasis, it is enough to show that the inverse image of any member of this
collection under \( f \) is open in \( X \). To this end, take any two real numbers \( a \) and \( b \) in \([0, 1)\) and \((0, 1]\), respectively. Our task is to show that \( f^{-1}((a, 1]) \) and \( f^{-1}([0, b)) \) are open in \( X \). First, note that \( f(x) < b \) iff \( x \in O_t \) for some \( t \in D \) with \( t < b \). It follows that

\[
f^{-1}([0, b)) = \bigcup \{O_t : b > t \in D\},
\]

and hence \( f^{-1}((0, b)) \) is an open subset of \( X \). Next, given the nested structure of \( O_d \)s, we note that \( f(x) \leq a \) iff \( x \in O_s \) for every \( s \in D \) with \( s > a \). It follows that

\[
f^{-1}([0, a]) = \bigcap \{O_s : a < s \in D\}.
\]

Obviously, the right-hand side is contained in \( \bigcap \{\text{cl}(O_t) : a < t \in D\} \). Conversely, if \( x \) belongs to this latter set, and \( s \) is any dyadic rational number in \([0, 1]\) with \( a < s \), we can choose a \( t \in D \) with \( a < t < s \) (since \( D \) is dense in \([0, 1]\)), and thus see that \( x \in \text{cl}(O_t) \subseteq O_s \). This shows that these two intersections are one and the same, that is,

\[
f^{-1}([0, a]) = \bigcap \{\text{cl}(O_t) : a < t \in D\}.
\]

Thus, \( f^{-1}([0, a]) \) is a closed set in \( X \), which implies that \( f^{-1}((a, 1]) \) is open in \( X \), as we sought. Our proof is now complete.

### 2.3 Consequences of Urysohn’s Lemma

Urysohn’s Lemma is the key toward discovering a number of far reaching properties of normal spaces. We consider some of the most important of these properties in this subsection.

#### 2.3.1 The Tietze Extension Theorem

Recall that every Lipschitz map on a metric subspace \( S \) of a given metric space \( X \) can be extended to a Lipschitz map that is defined on the entire \( X \). (This is the Kirzbraun-McShane Extension Theorem we proved in Section 2.2 of Chapter 2.) When \( X \) is a topological, but not metric, space, we cannot talk about the Lipschitz property, and hence such a theorem becomes out of place. However, we can still ask if it is possible to carry out such an extension for continuous real maps defined on a subset \( S \) of \( X \). This is certainly not true (even when \( X \) is metrizable) if we do not put any conditions on \( S \) and \( X \). (For instance, the map \( x \mapsto 1/x \) on \( \mathbb{R}_{++} \) cannot be continuously extended to \( \mathbb{R}_{++} \).) But, as shown by Heindrich Tietze in 1915, Urysohn’s Lemma can be used to show that all goes well when \( S \) is closed and \( X \) is normal.
The Tietze Extension Theorem. Let $X$ be a normal space, and $S$ a non-empty closed subset of $X$. For any real numbers $a$ and $b$ with $a \leq b$, and any continuous map $f : S \rightarrow [a,b]$, there is a continuous map $F : X \rightarrow [a,b]$ with $F|_S = f$.

To facilitate the proof of this result, we first establish an auxiliary approximation lemma.

**Lemma 2.4.** Let $X$ be a normal space, and $S$ a nonempty closed subset of $X$. For each $g \in C_b(S)$, there is an $h \in C_b(X)$ such that

$$|h(x)| \leq \frac{1}{3}K \quad \text{for every } x \in X$$

and

$$|g(x) - h(x)| \leq \frac{2}{3}K \quad \text{for every } x \in S,$$

where $K := \sup_{x \in S} |g(x)|$.

**Proof.** If $K = 0$, choosing $h$ as the zero function on $X$ settles the claim, so we assume $K > 0$. Put $A := g^{-1}[-K, -\frac{1}{3}K]$ and $B := g^{-1}[\frac{1}{3}K, K]$, which are disjoint closed subsets of $S$, and hence of $X$. By Urysohn’s Lemma, there is a continuous map $F : X \rightarrow [0,1]$ such that $F|_A = 0$ and $F|_B = 1$. We define

$$h := \frac{2}{3}K(F - \frac{1}{2}).$$

Then, $|h(x)| \leq \frac{2}{3}K|F(x) - \frac{1}{2}| \leq \frac{1}{3}K$ for every $x \in X$. Now take any $x$ in $S$. We wish to show that $|g(x) - h(x)| \leq \frac{2}{3}K$. If $x \in A$, then $h(x) = -\frac{1}{3}K$ and $g(x) \in [-K, -\frac{1}{3}K]$, so the sought inequality holds. If $x \in B$, an analogous argument applies. Finally, if $x \in S \setminus (A \cup B)$, we have $g(x) \in [-\frac{1}{3}K, \frac{1}{3}K]$, so since $|h(x)| \leq \frac{1}{3}K$, we again reach our target inequality.

**Proof of the Tietze Extension Theorem.** Let us first prove the result for $a = -1$ and $b = 1$. In this case, $\sup_{x \in S} |f(x)| \leq 1$, so applying Lemma 2.4 inductively, we obtain a sequence $(h_0, h_1, \ldots)$ in $C_b(X)$ such that

$$|h_m(x)| \leq \frac{1}{3} \left(\frac{2}{3}\right)^m \quad \text{for every } x \in X$$

and

$$|f(x) - (h_0 + \cdots + h_m)(x)| \leq \left(\frac{2}{3}\right)^{m+1} \quad \text{for every } x \in S$$

for each $m = 1, 2, \ldots$. (Here, $h_0$ is obtained from this lemma by setting $g = f$, and $h_m$ is obtained from it by setting $g = f - (h_0 + \cdots + h_{m-1})$ for each $m \geq 1$.) We now set $F := h_0 + h_1 + \cdots$. In view of the first set
of inequalities above, we can invoke the Weierstrass $M$-Test – recall Section 1.5 of Chapter 2 – to conclude that $F$ is a continuous function on $X$. As $|F(x)| \leq \frac{1}{3}(1 + \frac{2}{3} + \left(\frac{2}{3}\right)^2 + \cdots) = 1$ for each $x \in X$, we also have $F(X) \subseteq [-1, 1]$. Finally, from the second set of inequalities, we have $F|_S = f$, as we sought.

We now know that the Tietze Extension Theorem is valid when $[a, b] = [-1, 1]$. Next, take any real numbers $a$ and $b$ with $a < b$, and fix any homeomorphism $\varphi$ from $[a, b]$ onto $[-1, 1]$ with $\varphi(a) = -1$ and $\varphi(b) = 1$. Applying what we have just established to the map $\varphi \circ f$ furnishes a continuous $G : X \to [-1, 1]$ with $G|_S = \varphi \circ f$. Then, setting $F := \varphi^{-1} \circ G$ completes the proof of the Tietze Extension Theorem.

Remark. We can easily extend Tietze’s Theorem to the context of vector-valued maps by applying the original theorem componentwise. To wit, let $S$, $X$, $a$ and $b$ be as in Tietze’s Extension Theorem, and for any positive integer $n$, take any map $f$ in $C(S, [a, b]^n)$. For each $i = 1, \ldots, n$, put $f_i := \text{proj}_i \circ f$ and apply Tietze’s Theorem to find an $F_i \in C(X, [a, b])$ with $F_i|_S = f_i$. Then, $F : X \to [a, b]^n$, defined by $F(x) := (F_1(x), \ldots, F_n(x))$, is a continuous map with $F|_S = f$.

2.3.2 “Urysohn’s Lemma” for Locally Compact Spaces

As a locally compact Hausdorff space need not be normal (Example 2.7), neither Urysohn’s Lemma nor the Tietze Extension Theorem applies to such spaces. This is a shame because such spaces are encountered routinely in functional analysis and related fields. Fortunately, however, slightly weaker, but still powerful, forms of these results (in which the “closedness” hypotheses are strengthened to “compactness” hypotheses) are valid for locally compact Hausdorff spaces as well.

Let us begin with a preliminary observation (compare with Proposition 2.1).

Lemma 2.5. Let $X$ be a locally compact Hausdorff space. For every compact subset $C$ of $X$ and every open subset $O$ of $X$ with $C \subseteq O$, there is an open subset $U$ of $X$ such that $\text{cl}(U)$ is compact, and

$$C \subseteq U \subseteq \text{cl}(U) \subseteq O.$$ 

Proof. Take any compact subset $C$ of $X$. If $C$ is empty, our assertion becomes trivial, so we assume that it is not empty. For each $x$ in $C$, we use Proposition 4.1 of Chapter 7 to find an open subset $U_x$ of $X$ such that $\text{cl}(U_x)$ is compact and $x \in U_x \subseteq \text{cl}(U_x) \subseteq O$. Then, $\{U_x : x \in C\}$ is an open cover
of $C$, so, as $C$ is compact, there are finitely many points $x_1, \ldots, x_k$ in $C$ such that $\{U_{x_1}, \ldots, U_{x_k}\}$ covers $C$. It is readily checked that $U := U_{x_1} \cup \cdots \cup U_{x_k}$ satisfies the required properties.

The following takes $X$ as a locally compact Hausdorff space in the statement of Urysohn’s Lemma at the cost of taking $A$ in that statement as compact.

**Theorem 2.6.** Let $X$ be a locally compact Hausdorff space, and $A$ and $B$ two closed subsets of $X$. If $A$ is compact, there is a continuous map $f : X \to [0, 1]$ such that $f|_A = 0$ and $f|_B = 1$.

**Proof.** Assume that $A$ is compact, and use Lemma 2.5 to find an open subset $U$ of $X$ such that $\text{cl}(U)$ is compact and $A \subseteq U \subseteq \text{cl}(U) \subseteq X \setminus B$. Notice that $A$ and $\partial U$ are disjoint closed sets in $X$, and hence in $\text{cl}(U)$. Since $\text{cl}(U)$ is a compact Hausdorff space (as a subspace of $X$), it is normal, so we may apply Urysohn’s Lemma to find a continuous $g : \text{cl}(U) \to [0, 1]$ such that $g|_A = 0$ and $g|_{\partial U} = 1$. Now define $f : X \to [0, 1]$ as

$$f(x) := \begin{cases} 
  g(x), & \text{if } x \in \text{cl}(U), \\
  1, & \text{if } x \in X \setminus U.
\end{cases}$$

Clearly, $f|_A = 0$ and $f|_B = 1$. Furthermore, $f$ is continuous by the Pasting Lemma.

Finally, we derive a “Tietze Extension Theorem” for locally compact Hausdorff spaces.

**Theorem 2.7.** Let $X$ be a locally compact Hausdorff space, and $S$ a non-empty compact subset of $X$. For any $f \in C(S, [0, 1])$, there is an $F \in C(X, [0, 1])$ with $F|_S = f$.

**Proof.** By Lemma 2.5, there is an open subset $U$ of $X$ such that $\text{cl}(U)$ is compact and $S \subseteq U \subseteq \text{cl}(U)$. Since $\text{cl}(U)$ is a compact Hausdorff space (as a subspace of $X$), it is normal, so we may apply the Tietze Extension Theorem to find a continuous $g : \text{cl}(U) \to [0, 1]$ such that $g|_S = f$. Proof is completed in a way analogous to the proof of the previous theorem.

### 2.3.3 The Urysohn Embedding Theorem

Let $X$ be a topological space. A set $F \subseteq C(X)$ is said to separate points from closed sets in $X$ if for every $x \in X$ and open neighborhood $O$ of $x$
in $X$, there is an $f \in \mathcal{F}$ such that $f(x) \neq 0 = f|_{X\setminus O}$. As we will elaborate in the next section, this notion is quite important. For the moment, however, we will use it only to streamline the present discussion.

In general, there may not be a collection of continuous real maps that separates points from closed sets in a topological space. (This is true even for regular spaces!) Or, it may well be that we need too big of a set of continuous real maps to do this. In the case of second-countable regular spaces, however, the situation is quite pleasant:

**Lemma 2.8.** For any second-countable regular space $X$, there exists a countable subset $\mathcal{F}$ of $C(X, [0, 1])$ that separates points from closed sets in $X$.

**Proof.** Let $\mathcal{B}$ be a countable basis for the topology of $X$, and enumerate this set as $\{B_1, B_2, \ldots\}$. Let $T$ stand for the set of all $(k, l) \in \mathbb{N}^2$ such that $\text{cl}(B_l) \subseteq B_k$. In view of Proposition 2.3, for every $(k, l) \in T$, we may apply Urysohn’s Lemma to find a continuous $f_{k,l} : X \to [0, 1]$ with $f_{k,l}|_{\text{cl}(B_l)} = 1$ and $f_{k,l}|_{X \setminus B_k} = 0$. Let us put $\mathcal{F} := \{f_{k,l} : (k, l) \in T\}$ which is a countable subset of $C(X, [0, 1])$. We claim that $\mathcal{F}$ separates points from closed sets in $X$. To prove this, take any $x \in X$ and any $O \in \mathcal{O}_X(x)$. Clearly, there is a positive integer $k$ with $x \in B_k \subseteq O$. In turn, by Proposition 1.1, there is a $U \in \mathcal{O}_X$ with $x \in U \subseteq \text{cl}(U) \subseteq B_k$. Since $\mathcal{B}$ is a basis for $\mathcal{O}_X$, there must be a positive integer $l$ such that $x \in B_l \subseteq U$. But then, $\text{cl}(B_l) \subseteq \text{cl}(U) \subseteq B_k$, that is, $(k, l) \in T$, and hence $f_{k,l}(x) = 1$ and $f_{k,l}|_{X \setminus O} = 0$ (since $B_k \subseteq O$).

We use this lemma to obtain the following insight about the structure of second-countable regular spaces. (Note. This result generalizes Theorem 3.2 of Chapter 2.)

**The Urysohn Embedding Theorem.** Every second-countable regular space $X$ can be embedded in the Hilbert cube.

**Proof.** By Lemma 2.8, there is a countable subset $\mathcal{F}$ of $C(X, [0, 1])$ that separates points from closed sets in $X$. Let us enumerate this collection as $\{f_1, f_2, \ldots\}$. We define $\Phi : X \to [0, 1]^\infty$ by

$$\Phi(x) := (f_1(x), f_2(x), \ldots).$$

It is easy to check that this map is injective. (Let $x$ and $y$ be two distinct points in $X$, and pick an $O \in \mathcal{O}_X(x)$ that does not contain $y$. Since $\mathcal{F}$ separates points from closed sets, there is a $k \in \mathbb{N}$ with $f_k(x) \neq 0 = f_k|_{X\setminus O}$. But then $f_k(x) \neq f_k(y)$, and hence, $\Phi(x) \neq \Phi(y)$.) Furthermore, using Proposition 5.4 of Chapter 4 and recalling that convergence on $[0, 1]^\infty$ is
coordinatewise, we can readily verify that $\Phi$ is continuous. We will complete our proof by showing that $\Phi^{-1}$ is a continuous map from $\Phi(X)$ onto $X$. In other words, we wish to prove that $\Phi$ is an open map from $X$ onto $\Phi(X)$. To this end, take any nonempty open subset $O$ of $X$. Let $y$ be a point in $\Phi(O)$, and put $x := \Phi^{-1}(y)$. Then, $x \in O$, so there is a positive integer $k$ with $f_k(x) > 0$ and $f_k|X \setminus O = 0$. We put $U := I_1 \times I_2 \times \cdots$ where $I_k := (0, 1]$ and $I_i := [0, 1]$ for each $i \in \mathbb{N} \setminus \{k\}$. Then, $U$ is an open set in the Hilbert cube with $y = \Phi(x) \in U \cap \Phi(X)$. Notice further that if $w \in U \cap \Phi(X)$, then $w = \Phi(z)$ for some $z \in X$ with $f_k(z) > 0$. Then, $z$ does not belong to $X \setminus O$, that is, $z \in O$, which means that $w \in \Phi(O)$. Conclusion: $y \in U \cap \Phi(X) \subseteq \Phi(O)$. In view of the arbitrary choice of $y$ in $\Phi(O)$, therefore, we may conclude that $\Phi(O)$ is open in $\Phi(X)$. In turn, in view of the arbitrary choice of $O$, we conclude that $\Phi$ is an open map from $X$ onto $\Phi(X)$, as we sought.

**Warning.** Regularity cannot be relaxed to the Hausdorff property in the Urysohn Embedding Theorem. For instance, $\mathbb{R}$, relative to the Smirnov topology, is a second-countable Hausdorff space, but it is not metrizable (as it is not regular).

**Remark.** The Hilbert cube may be replaced with $\ell_2$ in the Urysohn Embedding Theorem, that is, every second-countable regular space can be embedded in $\ell_2$. This can be proved either by using the method outlined in Exercise 3.10 of Chapter 2, or by defining $\Phi$ in the proof above by $\Phi(x) := (f_1(x), \frac{1}{2} f_2(x), \frac{1}{3} f_3(x), \ldots)$.

As an immediate corollary of the Urysohn Embedding Theorem, we see that every second-countable regular space is metrizable; this is the famous **Urysohn Metrization Theorem.** Since every metrizable space is regular, we thus obtain here a complete characterization of the metrizability of second-countable spaces: A second-countable topological space is metrizable iff it is regular.

Here are two cute applications of the Urysohn Metrization Theorem.

**Example 2.8.** Every second-countable locally compact Hausdorff space is metrizable. (Proof. Recall Corollary 1.2.)

**Example 2.9.** Let us endow $\mathbb{Q}$ with the topology generated by the basis $\{[a, b) : a, b \in \mathbb{Q}\}$. (Naturally, we call this topology the *Sorgenfrey topology* on $\mathbb{Q}$.) Let us denote the resulting topological space by $X$. Obviously, $X$ is second-countable, while the argument we gave in Example 2.2 shows that it is normal as well. Thus, by the Urysohn Metrization Theorem, $X$ is
metrizable. Moreover, evidently, $X$ has no isolated points. We may thus apply Sierpiński’s Theorem – see Exercise 5.3 of Chapter 7 – to conclude that $X$ is homeomorphic to $\mathbb{Q}$. This is quite surprising, because $X$ surely has more open sets than $\mathbb{Q}$. (Every open subset of $\mathbb{Q}$ is open in $X$, but, say, $[0,1] \cap \mathbb{Q}$ is open in $X$ but not in $\mathbb{Q}$.) And yet, apparently, $X \cong \mathbb{Q}$!

2.3.4 Application: Metrizability of the Unit Ball of the Dual Space$^2$

Let $X$ be a normed linear space, and as usual, denote by $X^*$ the normed linear space of all continuous linear functionals on $X$, where the norm we use on $X^*$ is the operator norm. We have seen in Section 3.2.4 of Chapter 7 that endowing $X^*$ with the topology of pointwise convergence (that is, the weak*-topology), ensures that the closed unit ball $B_{X^*}$ of $X^*$ is compact (Alaoglu’s Theorem). This is an advantage of topologizing $X^*$ by the weak*-topology, for $B_{X^*}$ is not compact relative to the norm topology of $X^*$ unless $X$ is finite-dimensional. On the other hand, the norm topology on $X^*$ is of course metrizable, while weak*-topology is not (again, unless $X$ is finite-dimensional). However, as we will now prove, we get the best of the two worlds on $B_{X^*}$ as long as the underlying normed linear space is separable. Our proof is based on the Urysohn Metrization Theorem.

**Proposition 2.9.** Let $X$ be a separable normed linear space. Then, $B_{X^*}$ is a compact metrizable space relative to the topology of pointwise convergence.

**Proof.** As the topological product of Hausdorff spaces is Hausdorff, and the Hausdorff property is hereditary, Alaoglu’s Theorem says that $B_{X^*}$ is a compact Hausdorff space. Now, note that

$$\left\{ \{ f \in X^* : |f(x_i)| < \epsilon \text{ for all } i \in [k] \} : \epsilon > 0, x_1, \ldots, x_k \in X, k \in \mathbb{N} \right\}$$

is a basis for the topology of pointwise convergence on $X^*$. But then, where $Y$ is a countable dense subset of $X$,

$$\left\{ \{ f \in X^* : |f(x_i)| < \epsilon \text{ for all } i \in [k] \} : \epsilon \in \mathbb{Q}_{++}, x_1, \ldots, x_k \in Y, k \in \mathbb{N} \right\}$$

is a countable basis for the topology of pointwise convergence on $X^*$. (Why?) As second-countability is hereditary, therefore, $B_{X^*}$ is a second-countable compact Hausdorff space. Applying the Urysohn Metrization Theorem completes our proof.

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$^2$This section presumes familiarity with Section 3.2.4 of Chapter 7.
Remark. We have used the Urysohn Metrization Theorem above mainly for illustrative reasons. Indeed, it is possible to give a more direct proof for the metrizability part of Proposition 2.9 by using the metric $d$ on $X^*$ given by $d(f, g) := \sum_1^\infty 2^{-i} |f(x_i) - g(x_i)|$, where $\{x_1, x_2, \ldots\}$ is a dense subset of $B_{X^*}$.

Exercises

2.1. Prove that a closed subspace of a normal space is normal.

2.2. Prove Proposition 2.1.

2.3. Prove that the Moore plane is not normal.

2.4. Let $X$ and $Y$ be topological spaces. Prove: If $X$ is normal and there is a continuous and closed surjection from $X$ onto $Y$, then $Y$ is normal as well.

2.5. (Converse of Urysohn’s Lemma) Let $X$ be a $T_1$-space such that for any two closed subsets $A$ and $B$ of $X$, there is an $f \in C(X)$ with $f|_A = 0$ and $f|_B = 1$. Show that $X$ is normal.

2.6. Prove that a Lindelöf space is normal iff it is regular. (Hint: Recall Exercise 1.27 of Chapter 7, and imitation the proof of Proposition 2.3.)

2.7. Let $X$ be a $T_1$-space. We say that $X$ is completely normal if for every two closed sets $A$ and $B$ in $X$ with $A \cap \text{cl}(B) = \emptyset = \text{cl}(A) \cap B$, there are disjoint open sets $O$ and $U$ in $X$ with $A \subseteq O$ and $B \subseteq U$. Show that $X$ is completely normal iff every subspace of $X$ is normal.

2.8. Let $X$ be a normal space, and $A$ and $B$ two nonempty disjoint closed $G_\delta$-sets in $X$. Prove that there exists an $f \in C(X, [0, 1])$ with $f^{-1}(0) = A$ and $f^{-1}(1) = B$.

2.9. Let $(X, \rightarrow)$ be a loset. Prove that $X$ is a normal space relative to the $\rightarrow$-order topology.

2.10. (Partition of Unity) Let $X$ be a normal space, and for any integer $m \geq 2$, let $\{O_1, \ldots, O_m\}$ be an open cover of $X$. We prove in this exercise (by induction) that there are maps $f_1, \ldots, f_m$ in $C(X, [0, 1])$ such that $f_1 + \cdots + f_m = 1$ and $f_i|_{X \setminus O_i} = 0$ for each $i = 1, \ldots, m$.

   a. Use Urysohn’s Lemma to settle the claim for the case $m = 2$.

   b. Assume that our claim holds when $m$ equals an arbitrarily fixed integer $k \geq 2$, and let $\{O_1, \ldots, O_{k+1}\}$ be an open cover of $X$. Apply the induction hypothesis to find $g_1, \ldots, g_k \in C(X, [0, 1])$ such that $g_1 + \cdots + g_k = 1$, $g_i|_{X \setminus O_i} = 0$ for each $i = 1, \ldots, k-1$, and $g_k|_{X \setminus (O_1 \cup \cdots \cup O_k)} = 0$. Next, use part (a) to find two maps $h_1$ and $h_2$ in $C(X, [0, 1])$ with $h_1 + h_2 = 1$, $h_1|_{X \setminus (O_1 \cup \cdots \cup O_k)} = 0$, and $h_2|_{X \setminus O_{k+1}} = 0$.

   c. Put $f_i := h_1 g_i$ for each $i = 1, \ldots, k$ and $f_{k+1} := h_2 g_k$. Check that $f_1 + \cdots + f_{k+1} = 1$ and $f_i|_{X \setminus O_i} = 0$ for each $i = 1, \ldots, k+1$.  

20
2.11. Prove the following derivative of the Tietze extension Theorem: Let \( X \) be a normal space, and \( S \) a nonempty closed subset of \( X \). For any \( f \in C(S) \), there is an \( F \in C(X) \) with \( F|_S = f \).

2.12. Let \( S \) be a nonempty closed subset of a normal space \( X \). For any positive integer \( m \), suppose that \( Y_m \) is a Hausdorff space with the following property: Every continuous \( f : S \to Y_m \) has a continuous extension defined on \( X \). Now let \( N \) be a nonempty subset of \( \mathbb{N} \), and \( Y \) the topological product of \( \{ Y_i : i \in N \} \). Show that every continuous \( \varphi : S \to Y \) has a continuous extension defined on \( X \).

2.13. Prove that every second-countable regular space can be embedded in \( \ell_2 \).

2.14. Let \( X \) be a compact metric space and \( Y \) a Hausdorff space. Prove: \( f(X) \) is a metrizable subspace of \( Y \) for any \( f \in C(X,Y) \).

3 Completely Regular Spaces

First Impressions

One consequence of Urysohn’s Lemma is that a \( T_1 \)-space is normal iff for every disjoint closed subsets \( A \) and \( B \) of \( X \) there is an \( f \in C(X) \) such that \( f|_A = 0 \) and \( f|_B = 1 \). This characterization opens up other ways of defining separation properties for topological spaces. For instance, by asking \( A \) and \( B \) to be singletons here, we arrive at the class of topological spaces \( X \) such that \( C(X) \) separates the points of \( X \) in the sense that for every distinct \( x, y \in X \) there is an \( f \in C(X) \) with \( f(x) \neq f(y) \). The set \( C(X) \) of all continuous real maps on any one member \( X \) of this class (which contains the class of all normal spaces) is fairly rich; in particular \( C(X) \) contains maps other than the constant ones.

Things get even more interesting if we don’t go insofar as restricting both \( A \) and \( B \) above to be singletons, but ask this for only one of them. When \( X \) is as such, \( C(X) \) not only separates the points of \( X \), but it separates points from closed sets in \( X \), that is, for any \( x \in X \) and \( C \in \mathcal{C}_X \) with \( x \in X\setminus C \), there is an \( f \in C(X) \) with \( f(x) \neq 0 = f|_C \). When they also satisfy the \( T_1 \)-axiom, such spaces are of great interest and bear a special name.

Definition. Let \( X \) be a topological space. We say that \( X \) (or the topology of \( X \)) is completely regular if it is a \( T_1 \)-space and for every closed set \( C \) in \( X \) and \( x \in X\setminus C \), there is a continuous real map \( f \) on \( X \) such that \( f(x) = 1 \) and \( f|_C = 0 \). (In this case, we may refer to \( X \) simply as a completely regular space.)
Warning. Completely regular spaces sometimes also go by the name Tychonoff spaces.

Remark. The choice of the numbers 0 and 1 in this definition is only for concreteness; all that matters is that these numbers be distinct. (Indeed, suppose $X$ is completely regular, and take any distinct real numbers $a$ and $b$. Then, for any $C \subset X$ and $x \in X \setminus C$, there is an $f \in \mathcal{C}(X)$ with $f(x) = 1$ and $f|_C = 0$, so $g := (a - b)f + b$ is a continuous real map on $X$ with $g(x) = a$ and $g|_C = b$.)

More importantly, it is without loss of generality to posit in the context of the above definition that the range of $f$ is contained in $[0, 1]$. (Indeed, suppose $X$ is completely regular, and take any nonempty $C \subset X$ and $x \in X \setminus C$. Then, there is an $f \in \mathcal{C}(X)$ with $f(x) = 1$ and $f|_C = 0$, so $g := \max\{\min\{f, 1\}, 0\}$ is a continuous $[0, 1]$-valued map on $X$ such that $g(x) = 1$ and $g|_C = 0$.)

By definition, complete regularity of $X$ implies that separates points from closed sets in $X$. For $T_1$-spaces, the converse of this implication holds as well.

Lemma 3.1. Let $X$ be a $T_1$-space. Then, $X$ is completely regular if, and only if, $\mathcal{C}(X)$ separates points from closed sets in $X$.

Proof. We only need to prove the “if” part of the assertion. To this end, assume that $\mathcal{C}(X)$ separates points from closed sets in $X$, and take any nonempty $C \subset X$ and $x \in X \setminus C$. Then, $X \setminus C \in \mathcal{O}_X(x)$, so, by hypothesis, there is a $g \in \mathcal{C}(X)$ with $g(x) \neq 0 = g|_C$. But then, $f := g/g(x)$ is a continuous real map on $X$ with $f(x) = 1$ and $f|_C = 0$.

Regularity vs. Complete Regularity

We have seen so far that every regular space is Hausdorff, every Hausdorff space is $T_1$, and every $T_1$-space is $T_0$. Complete regularity adds one more layer to this hierarchy.

Proposition 3.2. Every completely regular space is regular.

Proof. Take any $C \subset X$ and $x \in X \setminus C$. By complete regularity, there is an $f \in \mathcal{C}(X)$ such that $f(x) = 1$ and $f|_C = 0$. Then, $f^{-1}((-\infty, \frac{1}{2}))$ and $f^{-1}((\frac{1}{2}, \infty))$ are disjoint open subsets of $X$, the first containing $C$ and the second containing $x$. 

22
The converse of Proposition 3.2 is false but, unfortunately, constructing a concrete example to this effect is somewhat involved. We do provide such a construction in the form of an exercise below, but you should note that most of the topological spaces that are encountered in practice are completely regular.

**Complete Regularity follows from Normality or Local Compactness**

As singletons are closed in a normal space, an immediate consequence of Urysohn’s Lemma is the following:

**Proposition 3.3.** *Every normal space is completely regular.*

Thus, we have the following hierarchy for topological spaces:

\[
\text{normality} \implies \text{complete regularity} \implies \text{regularity} \implies \text{Hausdorff condition} \implies T_1\text{-condition}
\]

Another set of examples of completely regular spaces are supplied by locally compact Hausdorff spaces. (Recall that a locally compact Hausdorff space need not be normal; see Example 2.7.)

**Proposition 3.4.** *Every locally compact Hausdorff space is completely regular.*

**Proof.** Apply Theorem 2.6.

**More Examples**

**Example 3.1.** Any zero-dimensional $T_1$-space – for instance, any discrete space – is sure to be completely regular. Indeed, if $X$ is such a space, for any closed set $C$ in $X$ and $x \in X \setminus C$, there is a clopen neighborhood $A$ of $x$ in $X$ with $A \cap C = \emptyset$, and the map $f : X \to \mathbb{R}$ defined by

\[
f(y) := \begin{cases} 
1, & \text{if } y \in A \\
0, & \text{otherwise,}
\end{cases}
\]

is continuous.

**Example 3.2.** Every metric space $X$ is completely regular. As all metric spaces are normal, this follows readily from Proposition 2.3. It is also easy to offer a direct proof as well. Indeed, for any nonempty closed proper subset
$C$ of $X$ and $x \in X \setminus C$, the map $\frac{\text{dist}(\cdot, C)}{\text{dist}(x, C)}$ is continuous and takes value $1$ at $x$ and $0$ on $C$.

**Example 3.3.** Every subspace $Y$ of a completely regular space $X$ is completely regular (that is, complete regularity is hereditary). To see this, take any $C \in \mathcal{C}_Y$ and $x \in Y \setminus C$. By definition of the subspace topology, there is an $S \in \mathcal{C}_X$ such that $C = S \cap Y$. (Notice that $x$ cannot belong to $S$.) Since $X$ is completely regular, therefore, there is an $f \in C(X)$ with $f(x) = 1$ and $f|_S = 0$. But then $g := f|_Y \in C(Y)$, and we have $g(x) = 1$ and $g|_C = 0$.

**A Characterization of Completely Regular Spaces**

There is a sense in which the topology of a completely regular space is determined entirely by the continuous (and bounded) real maps defined on that space. This is indeed one of the primary reasons for such spaces being of import to functional analysis.

To make things precise, let us revisit the notion of weak topologies – see Exercise 4.15 of Chapter 4 – albeit in the context of real maps. Let $\mathcal{F}$ be a nonempty collection of real maps on a nonempty set $X$. By definition, the weak topology on $X$ induced by $\mathcal{F}$ – we denote this topology by $\mathcal{O}(\mathcal{F})$ in this paragraph – is the smallest topology on $X$ with respect to which each $f \in \mathcal{F}$ is continuous. We can make this definition a bit more operational in the following manner. Let $\mathcal{I}$ stand for the set of all open intervals of the form either $(-\infty, a)$ or $(a, \infty)$. (Of course, the standard topology of $\mathbb{R}$ is generated by using $\mathcal{I}$ as a subbasis.) It is plain that every element of $\{f^{-1}(I) : I \in \mathcal{I} \text{ and } f \in \mathcal{F}\}$ is open relative to weak topology on $X$ induced by $\mathcal{F}$. Consequently, where $\mathcal{O}$ stands for the topology generated by using this collection as a subbasis, we have $\mathcal{O} \subseteq \mathcal{O}(\mathcal{F})$. But the inverse image of any open interval under $f \in \mathcal{F}$ is surely contained in $\mathcal{O}$, so each $f$ in $\mathcal{F}$ is continuous relative to the topology $\mathcal{O}$. It follows that $\mathcal{O} = \mathcal{O}(\mathcal{F})$.

Conclusion: The weak topology on $X$ induced by $\mathcal{F}$ is the topology generated by using as a subbasis the collection of all sets of the form $f^{-1}((-\infty, a))$ or $f^{-1}((a, \infty))$, where $a \in \mathbb{R}$ and $f \in \mathcal{F}$. We will use this characterization of $\mathcal{O}(\mathcal{F})$ below for the case where $\mathcal{F} = \mathcal{C}_b(X)$.

Here is the sense in which completely regular topologies are determined by the real maps that they render continuous.

**Proposition 3.5.** A $T_1$-space $X$ is completely regular if, and only if, the topology of $X$ is the weak topology on $X$ induced by $\mathcal{C}_b(X)$.

**Proof.** Let us denote by $\mathcal{O}^w_X$ the weak topology on $X$ induced by $\mathcal{C}_b(X)$. It is obvious that $\mathcal{O}^w_X \subseteq \mathcal{O}_X$. Let us now assume that $X$ is completely
regular, and establish the converse containment. Take any $O$ in $\mathcal{O}_X$. If $O$ is empty, then it obviously belongs to $\mathcal{O}_X^w$, so assume that $O \neq \emptyset$. For any $x$ in $O$, complete regularity of $X$ ensures that there is an $f_x \in \mathbf{C}_b(X)$ such that $f_x(x) = 1$ and $f_x|_{X \setminus O} = 0$. Put $O_x := f_x^{-1}((0,\infty))$, and note that $O_x \in \mathcal{O}_X^w$, for each $x \in O$. Besides, $x \in O_x \subseteq O$ for every $x \in O$, so $O = \bigcup \{O_x : x \in O\}$. Thus, being the union of a collection of open sets, $O$ is open, relative to the weak topology on $X$ induced by $\mathbf{C}_b(X)$. Conclusion: $\mathcal{O}_X^w = \mathcal{O}_X$.

Conversely, assume that the topology of $X$ is the weak topology on $X$ induced by $\mathbf{C}_b(X)$. We will prove that $\mathbf{C}_b(X)$ separates points from closed sets in $X$. So, take any $x \in X$ and $O \in \mathcal{O}_X(x)$. By the discussion preceding the statement of the present proposition, there exist finitely many $f_1,\ldots,f_k \in \mathbf{C}_b(X)$ and $I_1,\ldots,I_k \in \mathcal{I}$ (where $\mathcal{I}$ is as defined above) such that $x \in f_1^{-1}(I_1) \cap \cdots \cap f_k^{-1}(I_k) \subseteq O$. It is actually without loss of generality to assume that each $I_i$ is of the form $(a_i,\infty)$ for some real number $a_i$. (If $I_i = (-\infty,a_i)$, then we would replace $f_i$ by $-f_i$, and use the fact that $f_i^{-1}(I_i) = (-f_i)^{-1}((a_i,\infty))$.) Now, for each $i = 1,\ldots,k$, define the real map $g_i$ on $X$ by $g_i(z) := \max\{f_i(z) - a_i,0\}$. Then, $g_i^{-1}(\mathbb{R}_{++}) = f_i^{-1}(I_i)$ for each $i$, so

$$x \in \bigcap_{i=1}^k g_i^{-1}(\mathbb{R}_{++}) \subseteq O. \tag{3}$$

Finally, we define $f \in \mathbf{C}_b(X)$ by $f := \prod_{i=1}^k g_i$. It follows from (3) that $f(x) \neq 0$. Moreover, $f|_{X \setminus O} = 0$, because $f \geq 0$, so if $y \in X$ satisfies $f(y) \neq 0$, then $g_i(y) > 0$ for all $i = 1,\ldots,k$, and (3) implies $y \in O$. In view of the arbitrary choice of $x$ and $O$, we conclude that $\mathbf{C}_b(X)$ separates points from closed sets in $X$, that is, $X$ is completely regular.

**Inheritance Properties of Completely Regular Spaces**

We have already seen that complete regularity is hereditary. We now show that this property also behaves well in terms of the process of forming topological products.

**Proposition 3.6.** Let $I$ be a nonempty set, $X_i$ a topological space for each $i \in I$, and $X$ the topological product of $\{X_i : i \in I\}$. Then, $X$ is completely regular if, and only if, $X_i$ is completely regular for each $i \in I$.

**Proof.** Suppose $X$ is completely regular, and take any $i \in I$. Let $C_i$ be a closed proper subset of $X_i$, and take any $x_i \in X_i \setminus C_i$. Fix an arbitrary element $x$ of $X$ with $x(i) = x_i$. We put $C := \text{proj}_i^{-1}(C_i)$, and note that
\( x \in X \setminus C \). By complete regularity of \( X \), therefore, there is an \( f \in C(X) \) such that \( f(x) = 1 \) and \( f|_C = 0 \). We define \( g : X_i \to X \) by
\[
g(z)(j) := \begin{cases} z, & \text{if } j = i \\ x(j), & \text{otherwise}, \end{cases}
\]
which is readily checked to be continuous. Now put \( h := f \circ g \), and note that \( h \in C(X_i) \) and \( h(x_i) = f(x) = 1 \). Moreover, for any \( z \in C_i \), we have \( g(z) \in C \), and hence \( f(g(z)) = 0 \), that is, \( h|_{C_i} = 0 \).

Conversely, suppose that \( X_i \) is completely regular for each \( i \in I \). Then, \( X \) is a \( T_1 \)-space (Proposition 1.3 of Chapter 5). Now take any \( C \in C_X \) and any \( x \in X \setminus C \). By definition of the product topology, there is a finite subset \( J \) of \( I \), and for each \( i \in I \), an open neighborhood \( O_i \) of \( x(i) \) in \( X_i \), such that \( x \in \bigcap_{i \in I} O_i \subseteq X \setminus C \) and \( O_i = X_i \) for all \( i \in I \setminus J \). Since \( X_j \) is completely regular, there is an \( f_j \in C(X_j) \) with \( f_j(x(j)) = 1 \) and \( f_j|_{X_j \setminus O_j} = 0 \), for every \( j \in J \). We define the real map \( f \) on \( X \) by
\[
f(y) := \prod_{j \in J} f_j(y(j)).
\]
It is readily checked that this function is continuous, while \( f(x) = 1 \) and \( f|_C = 0 \). In view of the arbitrary choice of \( C \) and \( x \), we may thus conclude \( X \) is completely regular.

As you might expect, things do not go so well with respect to the process of forming quotients. Indeed, the quotient of a completely regular space \( X \) with respect to a closed equivalence relation on \( X \) need not even be Hausdorff in general.

**The Embedding Lemma**

The Urysohn Embedding Theorem says that every second-countable regular space can be embedded in the Hilbert cube. But can we say something along these lines without assuming that the space at hand has a countable basis? There is a very nice answer to this question, to which we now turn.

We begin with the following apparently technical, but surprisingly useful, result.

**The Embedding Lemma.** Let \( X \) be a \( T_1 \)-space, and \( F \) a collection of continuous real functions on \( X \) that separates points from closed sets in \( X \). Then, the map \( \Psi : X \to \mathbb{R}^F \), defined by \( \Psi(x)(f) := f(x) \) for all \( f \in F \), is
an embedding (relative to the topology of pointwise convergence on \( \mathbb{R}^\mathcal{F} \)). If
\( f(X) \subseteq [0,1] \) for each \( f \in \mathcal{F} \), then \( \Psi \) embeds \( X \) in \([0,1]^\mathcal{F}\).

**Proof.** For any net \((x_\alpha)\) in \( X \) with \( x_\alpha \rightarrow x \) for some \( x \in X \), we have
\( \Psi(x_\alpha)(f) = f(x_\alpha) \rightarrow f(x) = \Psi(x)(f) \) for each \( f \in \mathcal{F} \) (since \( \mathcal{F} \subseteq C(X) \)), so \( \Psi(x_\alpha) \rightarrow \Psi(x) \) (relative to the topology of pointwise convergence on \( \mathbb{R}^\mathcal{F} \)).

In view of Proposition 5.4 of Chapter 4, therefore, we may conclude that \( \Psi \) is continuous. On the other hand, since \( X \) is a \( T_1 \)-space and \( \mathcal{F} \) separates points from closed sets in \( X \), for every distinct \( x \) and \( y \) in \( X \), there is an \( f \in \mathcal{F} \) with \( f(x) \neq f(y) \). Thus: \( \Psi \) is a continuous bijection from \( X \) onto \( Y := \Psi(X) \). (Of course, we view \( Y \) as a subspace of \( \mathbb{R}^\mathcal{F} \) here.)

We are now to prove that \( \Psi \) is an open map. To this end, fix an arbitrary \( O \in \mathcal{O}_X \), and pick any \( \psi \in \Psi(O) \). We wish to find an open neighborhood \( V \) of \( \psi \) in \( Y \) that fits within \( \Psi(O) \). Let \( x := \Psi^{-1}(\psi) \), and notice that \( O \) is an open neighborhood of \( x \) in \( X \) so that there is a \( g \in \mathcal{F} \) with \( g(x) \neq 0 = g|_{X\setminus O} \) (because \( \mathcal{F} \) separates points from closed sets in \( X \)). Without loss of generality, let us assume that \( g(x) > 0 \). We then put
\[
U := \{ \varphi \in \mathbb{R}^\mathcal{F} : \varphi(g) > 0 \},
\]
which is an open subset of \( \mathbb{R}^\mathcal{F} \), and define \( V := U \cap Y \), which is open in \( Y \). Note that \( \psi \in V \), because \( \psi(g) = \Psi(x)(g) = g(x) > 0 \). Moreover, \( V \subseteq \Psi(O) \), because if \( \varphi \in V \), then \( \varphi = \Psi(y) \) for some \( y \in X \) and \( g(y) = \Psi(y)(g) = \varphi(g) > 0 \), so, as \( g|_{X\setminus O} = 0 \), we find that \( y \in O \), that is, \( \varphi \in \Psi(O) \).

Conclusion: \( \Psi(O) \) is open in \( Y \). In view of the arbitrary choice of \( O \), we may thus conclude that \( \Psi \) is an open map.

**The Tychonoff Embedding Theorem**

Let \( X \) be a completely regular space. Then, \( I := C(X,[0,1]) \) separates points from closed sets in \( X \). The Embedding Lemma thus says that \( X \) can be embedded in \([0,1]^I\). Conversely, for any nonempty (index) set \( I \), Proposition 3.6 ensures that \([0,1]^I \) is completely regular (relative to the product topology). Since complete regularity is hereditary, therefore, any nonempty subset of \([0,1]^I \) is completely regular as well. This establishes the following, rather unexpected, characterization of completely regular spaces which was obtained by Andrei Tychonoff in 1930.

**The Tychonoff Embedding Theorem.** A topological space is completely regular if, and only if, it can be embedded in \([0,1]^I \) for some nonempty (index) set \( I \).
In particular, we find that dropping the second-countability hypothesis in the Urysohn Embedding Theorem, but strengthening regularity to normality, leads us to the following:

**Corollary 3.7.** Every normal space can be embedded in \([0,1]^I\) for some non-empty (index) set \(I\).

**Exercises**

3.1. Prove that complete regularity is a topological invariant.

3.2. Let \((X,\succ)\) be a linearly ordered set. Show that \(X\) is completely regular relative to the \(\succ\)-order topology.

3.3. Let \(X\) be any nonempty set, fix an \(x_\ast\) in \(X\), and make \(X\) a topological space using the included point topology induced by \(x_\ast\). Is \(X\) completely regular?

3.4. Let \(X\) be a completely regular space. Show that a net \((x_\alpha)\) in \(X\) converges to a point \(x\) in \(X\) iff \(f(x_\alpha) \to f(x)\) for every continuous \(f \in C(X,[0,1])\).

3.5. Prove: A \(T_1\)-space \(X\) is completely regular iff every nonempty closed set in \(X\) equals \(\bigcap\{f^{-1}(0) : f \in \mathcal{F}\}\) for some \(\mathcal{F} \subseteq C(X)\).

*M3.6. (Mysior’s Example)* Put \(S := \mathbb{R} \times \mathbb{R}_+\), and pick any point \(z\) in \(\mathbb{R}^2 \setminus S\), say, \(z := (0,-1)\). Put \(T := S \cup \{z\}\). Now, for every real number \(a\), define

\[
V_a := \{a\} \times [0,2], \quad D_a := \{(t,t-a) : a \leq t \leq a+2\}, \quad \text{and} \quad O_a := V_a \cup D_a.
\]

Put \(B_1 := \{x : x_2 > 0\}\), and let \(B_2\) contain all sets of the form \(O_a \setminus C\), where \(a \in \mathbb{R}\) and \(C\) is any finite subset of \(O_a\) which does not contain \((a,0)\). On the other hand, let \(B_3\) contain all sets of the form \(\{z\} \cup U_m\), where \(m \in \mathbb{N}\) and \(U_m := \{x \in S : x_1 \geq m\}\). Finally, put \(B := B_1 \cup B_2 \cup B_3\).

a. Show that \(B_1 \cup B_2\) is a basis for a topology on \(S\), and \(B\) is a basis for a topology on \(T\). (Relative to the latter topology, each \(x\) with \(x_2 > 0\) is isolated, \(\{O_a \setminus C : (a,0) \notin C \subseteq O_a\text{ and }|C| < \infty\}\) is a local basis at \((a,0)\) for any real number \(a\), and \(B_3\) is a local basis at \(z\).)

b. Let us consider \(T\) as a topological space relative to the topology generated by \(B\). Show that every element of \(B_1 \cup B_2\) is clopen in \(T\). (So, \(S\) is a zero-dimensional space relative to the topology generated by \(B_1 \cup B_2\).) In turn, use this fact to prove that \(T\) is regular.

c. Take any real number \(a\). Show that if \(g \in C(S)\) and \(g(a,0) = 0\), then \(g(w) = 0\) for all but countably many \(w \in O_a\).

d. Take any real number \(a\) and \(g \in C(S)\). Show that if \(g(w) = 0\) for all but countably infinitely many \(w \in V_a \setminus \{(a,0)\}\), then \(g(a,0) = 0\).

e. Take any positive integer \(m\) and \(f \in C(S)\). Suppose that \(f(a,0) = 0\) for infinitely many \(a \in [m, m+1]\). Use what we have found in parts (c) and (d)
to prove that, for any integer \( k \geq m \), we have \( f(a, 0) = 0 \) for all but countably many \( a \in [k, k+1] \).

f. Take any \( f \in C(T) \) such that \( f|_{[0,1] \times \{0\}} = 0 \). Use what we have found in part (e) to show that \( f(z) = 0 \), and conclude that \( T \) is not completely regular.

4 Stone-Čech Compactification

The theory of completely regular spaces we have sketched above, especially the Embedding Lemma, brings us very close to a fundamental notion of compactification. We take up this opportunity here, and introduce this notion (which is a subject of research in topology even now).

First Impressions

Let \( X \) be a topological space, and \( Y \) a compactification of \( X \). (Recall that this means that \( Y \) is a compact Hausdorff space which admits \( X \) as a dense subspace.) A question that is often encountered (say, in functional analysis) is if it is possible to extend a given continuous and bounded real map on \( X \) to a continuous real map on \( Y \). In general, the answer is a resounding no. For instance, \([0, 1]\) is a compactification of \((0, 1)\), but the map \( t \mapsto \sin(1/t) \) on \((0, 1)\), which is continuous and bounded, cannot be extended to a continuous map on \([0, 1]\). Similarly, the one-point compactification of \((0, 1)\) does not satisfy this “extendability” property.

We will see below that there is actually a unique (up to homeomorphism) notion of compactification relative to which we always get a positive answer to our question (provided that \( X \) is completely regular). This notion was investigated, independently, by Marshall Stone and Eduard Čech in 1937, and thus bears the names of these two mathematicians.

Definition. Let \( X \) be a topological space. A Stone-Čech compactification of \( X \) is a compactification \( Y \) of \( X \) such that for every \( f \in C_b(X) \), there is a (unique) \( F \in C(Y) \) with \( F|_X = f \).

Warning. The uniqueness requirement here comes for free. For, if \( X \) is a dense subspace of a topological space \( Y \), then any \( f \in C(X) \) (or actually, any continuous map from \( X \) into a Hausdorff space) may have at most one continuous extension to \( Y \). Indeed, suppose \( F \) and \( G \) are two maps in \( C(Y) \) such that \( F|_X = f = G|_X \) and \( F(y) \neq G(y) \) for some \( y \in Y \). Then, pick any two disjoint open sets \( O \) and \( U \) in \( \mathbb{R} \) with \( F(y) \in O \) and \( G(y) \in U \), and pick any \( V \in \mathcal{O}_Y(y) \) so that \( F(V) \subseteq O \) and \( G(V) \subseteq U \).

\(^{3}\)The boundedness hypothesis is used here out of necessity. If there is a continuous extension of a map \( f \in C(X) \) to \( Y \), then, as \( Y \) is compact, that extension, and hence \( f \) itself, must be continuous.
As \( X \) is dense in \( Y \), it intersects \( V \) at some point \( x \in X \), so, as \( F(x) = f(x) = G(x) \), we find that \( O \cap U \neq \emptyset \), a contradiction.

A natural question is if every \( f \in C_b(X, \mathbb{Z}) \) can be extended to an \( F \in C_b(Y, \mathbb{Z}) \), when \( Y \) is a Stone-\u0160ech compactification of \( X \), and \( Z \) is a topological space. (By definition, we can do this for \( Z = \mathbb{R} \).) Unsurprisingly, we need to put some conditions on \( Z \) for this to happen. In particular, it turns out that all goes well if \( Z \) is a compact Hausdorff space.

**Proposition 4.1.** (Stone) Let \( Y \) be a Stone-\u0160ech compactification of a completely regular space \( X \), and \( Z \) a compact Hausdorff space. Then, for every continuous \( f : X \to \mathbb{Z} \), there is a (unique) continuous \( F : Y \to \mathbb{Z} \) with \( F|_X = f \).

**Proof.** Since \( Z \) is completely regular, the Tychono\u0160 Embedding Theorem says that there is a nonempty (index) set \( I \) and an embedding \( \Psi \) from \( Z \) into \( [0,1]^I \). We put \( g := \Psi \circ f \), and for each \( i \in I \), define \( g(i) : X \to [0,1] \) by

\[
g(i)(x) := g(x)(i).
\]

Thus, \( g(i) \) is the \( i \)-th component function of \( g \) – that is, it is the composition of the projection map \( \omega \to \omega(i) \) (on \( [0,1]^I \)) with \( g \) – and hence, it is continuous, for each \( i \in I \). Since \( Y \) is a Stone-\u0160ech compactification of \( X \), for every \( i \in I \), there is a \( G(i) \in C(Y) \) such that \( G(i)|_X = g(i) \). We now define \( G : Y \to \mathbb{R}^I \) by

\[
G(y)(i) := G(i)(y).
\]

Then, \( G \) is continuous (Example 1.12 of Chapter 5) and \( G|_X = g \). Now notice that

\[
G(Y) = G(\text{cl}(X)) \subseteq \text{cl}(G(X)) = \text{cl}(\Psi(f(X))) \subseteq \text{cl}(\Psi(Z)) = \Psi(Z)
\]

where the second statement follows from continuity of \( G \), and the last statement is true because \( \Psi(Z) \), being the continuous image of a compact space, is a compact, and hence closed, subset of \( [0,1]^I \). We may thus define the map \( F : Y \to Z \) by \( F := \Psi^{-1} \circ G \). As \( \Psi^{-1} \) is a continuous map from \( \Psi(Z) \) onto \( Z \), \( F \) is continuous. Moreover, \( F(x) = \Psi^{-1}(G(x)) = \Psi^{-1}(g(x)) = \Psi^{-1}(\Psi(f(x))) = f(x) \) for every \( x \in X \), that is, \( F|_X = f \).

**Existence of Stone-\u0160ech Compactifications**

Which sorts of topological spaces admit Stone-\u0160ech compactifications? It is easily seen that complete regularity is a necessary condition for this. Indeed, let \( X \) be a topological space for which there is at least one Stone-\u0160ech
compactification $Y$. Since $Y$ is a Hausdorff compact space, it is normal, and hence, completely regular. As complete regularity is hereditary, therefore, $X$ must itself be completely regular. We now prove that the converse of this observation also holds, that is, every completely regular space admits a Stone-Čech compactification. (This is yet another characterization of complete regularity!)

**Theorem 4.2.a.** (Tychonoff) Every completely regular space $X$ has a Stone-Čech compactification.

**Proof.** Since $X$ is completely regular, $\mathcal{F} := C(X, [0, 1])$ separates points from closed sets in $X$. By the Embedding Lemma, then, the map $\Psi : X \to [0, 1]^\mathcal{F}$, defined by $\Psi(x)(f) := f(x)$ for all $f \in \mathcal{F}$, is an embedding (relative to the product topology on $[0, 1]^\mathcal{F}$). Furthermore, by the Tychonoff Theorem, $[0, 1]^\mathcal{F}$ is a compact Hausdorff space. By Proposition 4.6 of Chapter 7, therefore, there is a compactification $Y$ of $X$ such that there is an embedding $\Lambda : Y \to [0, 1]^\mathcal{F}$ with $\Lambda|_X = \Psi$. We wish to show that $Y$ is a Stone-Čech compactification of $X$. To this end, fix an $f$ in $C_b(X)$, and put $K := \|f\|_\infty$. If $K = 0$, then $f$ equals 0 everywhere, and thus obviously have a continuous extension to $Y$. Assume, then, $K > 0$, and define $F : Y \to [0, 1]$ by

$$F(y) := K\Lambda(y) \left( \frac{1}{K}f \right).$$

Clearly, $F$ is a multiple of the composition of the projection map from $[0, 1]^\mathcal{F}$ into $[0, 1]$, defined by $\varphi \mapsto \varphi \left( \frac{1}{K}f \right)$, with $\Lambda$. As such, $F$ is a continuous map. Moreover,

$$F(x) = K\Lambda(x) \left( \frac{1}{K}f \right) = K\Psi(x) \left( \frac{1}{K}f \right) = f(x) \quad \text{for every } x \in X.$$

Thus: $F$ is a continuous extension of $f$ to $Y$.

**Historical Note.** Andrei Tychonoff has proved Theorem 4.2.a in 1930 along with the fact that a Hausdorff space has a compactification iff it is completely regular. By contrast, the related (independent) contributions of Čech (with due credit to Tychonoff) and Stone (without due credit to Tychonoff) appeared in 1937. The naming of this compactification concept thus seems inaccurate. For this reason, for instance, Dudley (2002) suggests that Stone-Čech compactification should be renamed as “Tychonoff-Čech compactification.”

**Uniqueness of Stone-Čech Compactifications**

Topologically speaking, there can be at most one Stone-Čech Compactification of a completely regular space. Put more precisely:
Theorem 4.2.b. Let $X$ be a completely regular space, and let $Y_1$ and $Y_2$ be two Stone-Čech compactifications of $X$. Then, $Y_1 \cong Y_2$.

Proof. Let $\iota_1 : X \hookrightarrow Y_1$ be the inclusion map from $X$ into $Y_1$, and define $\iota_2$ analogously. Since $\iota_2$ is a continuous map from $X$ into the compact Hausdorff space $Y_2$, and $Y_1$ is a Stone-Čech compactification of $X$, we may apply Proposition 4.1 to find a continuous $F : Y_1 \to Y_2$ with $F|_X = \iota_2$. Similarly, there is a continuous $G : Y_2 \to Y_1$ with $G|_X = \iota_1$. We put $H := G \circ F$, which is a continuous self-map on $Y_1$. Notice that $H(x) = x$ for each $x \in X$. Consequently, we may think of $H$ as a continuous extension of the map $x \mapsto x$ (from $X$ into $Y_1$) to $Y_1$. Obviously, $\text{id}_{Y_1}$ is also such an extension of that map. But since $Y_1$ is Hausdorff, there can be at most one extension of the map $x \mapsto x$ (from $X$ into $Y_1$) to $Y_1$. (Recall the Warning given above right after the definition of Stone-Čech compactification.) Thus: $H = \text{id}_{Y_1}$. As we can similarly show that $F \circ G = \text{id}_{Y_2}$, we conclude that $F$ is a homeomorphism between $Y_1$ and $Y_2$ (whose inverse is $G$).

We now know that every completely regular space $X$ has a unique (up to homeomorphism) Stone-Čech compactification. This compactification of $X$ is commonly denoted as $\beta X$ in the literature. Unfortunately, it is in general quite difficult (and often times impossible) to “compute” $\beta X$ even for fairly well-behaved $X$. But, as we shall see in the next section, this does not mean that the notion of Stone-Čech compactification is not of use in applications.

Remark. For any compact Hausdorff space $X$, we have $X \cong \beta X$.

Exercises

4.1. Show that $S^1$ is not homeomorphic to $\beta \mathbb{R}$.

4.2. Let $X$ be a completely regular space, and $Y$ a topological space with $X \subseteq Y \subseteq \beta X$. Show that $\beta Y = \beta X$.

4.3. Let $X$ and $Y$ be two completely regular spaces. Prove: For any $f \in C(X, Y)$, there is an $F \in C(\beta X, \beta Y)$ such that $F|_X = f$.

4.4. For any completely regular space $X$, prove that $X$ is connected if and only if $\beta X$.

4.5. Let $X$ be a locally compact Hausdorff space which is not compact. Let $\succ$ be the binary relation on $\mathcal{Y}$ such that $Y \succ Z$ if there is a continuous surjection $F : Y \to Z$ with $F|_X = \text{id}_X$. Prove:
   a. $(\mathcal{Y}, \succ)$ is a preordered set;
   b. Any one-point compactification of $X$ is a minimal element of $\mathcal{Y}$ with respect to $\succ$;
   c. $\beta X$ is a maximal element of $\mathcal{Y}$ with respect to $\succ$. 

32
4.6. Let $X$ be a completely regular space. Prove that $X$ is locally compact iff $X$ is open in $\beta X$.

5 Applications to Functional Analysis

5.1 Separability of $C(X)$

We have seen in Theorem 3.2 of Chapter 7 that $C(X)$ is separable for any compact metrizable space $X$. As an application of the theory of Stone-Čech compactification, we now prove the converse of this fact.

Let us begin by noting that, when $X$ is a normal space, separability of $C_b(X)$ implies the metrizability of $X$. To see this, let $X$ be a normal space, and define the map $\Psi : X \to \mathbb{R}^{C_b(X)}$ by $\Psi(x)(f) := f(x)$ for all $f \in C_b(X)$. As $C_b(X)$ separates points from closed sets in $X$ (by Urysohn’s Lemma), the Embedding Lemma says that $\Psi$ is an embedding (relative to the topology of pointwise convergence on $\mathbb{R}^{C_b(X)}$). As it is easy to check that the range of $\Psi$ is contained in the closed unit ball of $C_b(X)^*$ — that is, $\Psi(X) \subseteq B_{C_b(X)^*}$ — we may thus conclude that $X$ is homeomorphic to a subspace of $B_{C_b(X)^*}$ relative to the topology of pointwise convergence. Applying Proposition 2.9, therefore, we find:

**Proposition 5.1.** Let $X$ be a normal space such that $C_b(X)$ is separable. Then, $X$ is metrizable.

Next, we show that for any completely regular $X$, the metric spaces $C_b(X)$ and $C(\beta X)$ are closely related.

**Lemma 5.2.** $C_b(X)$ and $C(\beta X)$ are isometric for any completely regular space $X$.

**Proof.** Let $X$ be a completely regular space so that $\beta X$ exists. Then, for every $f$ in $C_b(X)$, there is a unique $f^*$ in $C(\beta X)$ such that $f^*|_X = f$ (by definition of $\beta X$). Consequently, we define the map $\Phi : C_b(X) \to C(\beta X)$ by $\Phi(f) := f^*$. Then, for any $f \in C_b(X)$,

$$
\|\Phi(f)\|_\infty = \sup_{y \in \beta X} |\Phi(f)(y)| = \sup_{x \in X} |\Phi(f)(x)| = \sup_{x \in X} |f(x)| = \|f\|_\infty,
$$

where the second equation follows from the fact that $X$ is dense in $\beta X$ and $\Phi(f)$ is a continuous real map on $\beta X$. But it is readily checked that

---

4This section presumes familiarity with Sections 3.2.2, 3.2.4 and 3.2.5 of Chapter 7.
\( \Phi \) is a linear map. It thus follows that \( \| \Phi(f) - \Phi(g) \|_\infty = \| \Phi(f - g) \|_\infty = \| f - g \|_\infty \) for any \( f, g \in \mathcal{C}_b(X) \), as we sought.

We are now ready to launch our attack on the main prize. Take any normal space \( X \); and assume that \( \mathcal{C}(X) \) is separable. Then, \( \mathcal{C}_b(X) \) is separable (Proposition 7.2 of Chapter 1), so by Proposition 5.1, \( X \) is metrizable. Moreover, by Lemma 5.2, \( \mathcal{C}(\beta X) \) is separable. But since \( \beta X \) is a compact Hausdorff space, it is normal, so we may apply Proposition 5.1 again, this time to conclude that \( \beta X \) is metrizable.

We will now prove that \( X \) is compact by showing that \( X = \beta X \). To derive a contradiction, suppose there is a \( y \) in \( \beta X \setminus X \). As \( \beta X \) is metrizable, there must then exist a sequence \( (x_m) \) in \( X \) such that \( x_m \to y \). Without loss of generality, we may presume that all terms of this sequence are distinct.

We put \( A := \{x_2, x_4, \ldots\} \) and \( B := \{x_1, x_3, \ldots\} \). Then, \( A \) and \( B \) are disjoint closed sets in \( X \), so we may apply Urysohn’s Lemma to find a continuous map \( f : X \to [0, 1] \) with \( f|_A = 0 \) and \( f|_B = 1 \). In turn, by definition of Stone-Čech compactification, there is a continuous \( F \in \mathcal{C}_b(\beta X) \) with \( F|_X = f \). But then \( f(x_{2m}) \to F(y) \) and \( f(x_{2m+1}) \to F(y) \), and this implies \( 0 = F(y) = 1 \), a contradiction.

We now know:

**Theorem 5.3.** Let \( X \) be a normal space. Then, \( \mathcal{C}(X) \) is separable if, and only if, \( X \) is a compact metrizable space.

### 5.2 Embedding \( \mathcal{C}(X) \) in \( \ell_\infty \)

Let \( X \) be a compact metrizable space. Then, we know that \( \mathcal{C}(X) \) is a separable metric space, so, by the Fréchet Embedding Theorem, it can be isometrically embedded in \( \ell_\infty \). In words, no matter what \( X \) is (so long as it is compact and metrizable), we can think of \( \ell_\infty \) containing a “copy” of \( \mathcal{C}(X) \) which is indistinguishable from \( \mathcal{C}(X) \) insofar as metric analysis is concerned. This is already pretty impressive, but we can in fact do much better. After all, both \( \mathcal{C}(X) \) and \( \ell_\infty \) are normed linear spaces, so it is natural to ask if \( \ell_\infty \) also contains a “copy” of \( \mathcal{C}(X) \) which is indistinguishable from \( \mathcal{C}(X) \) insofar as normed linear analysis is concerned.

To formalize this query, let us introduce the following concept:

**Definition.** Let \( X \) and \( Y \) be two normed linear spaces. A function \( f : X \to Y \) is said to be a **linear isometry** if it is linear and

\[
\| f(x) \|_Y = \| x \|_X \quad \text{for every } x \in X.
\]
(Here, of course, \( \| \cdot \|_X \) stands for the norm of \( X \), and \( \| \cdot \|_Y \) for that of \( Y \).) If there exists such a map, we say that “\( X \) can be linearly isometrically embedded in \( Y \).” If \( f \) is, in addition, surjective, we refer to it as a **linear isometric isomorphism**; and if there exists such a map, we say that “\( X \) and \( Y \) are linearly isometrically isomorphic.”

Due to its linearity, a linear isometry \( f \) from \( X \) into \( Y \) is indeed an isometry (in terms of the metrics induced by the norms of \( X \) and \( Y \)), so \( X \) and \( f(X) \) are indistinguishable from the viewpoint of metric analysis. Moreover, again due to the linearity of \( f \), \( f(X) \) is actually a linear subspace of \( Y \). Thus, as it is injective (because for any distinct \( x \) and \( y \) in \( X \), we have \( \| f(x) - f(y) \|_Y = \| f(x - y) \|_Y = \| x - y \|_X > 0 \)), \( f \) is an invertible linear operator from \( X \) onto the linear subspace \( f(X) \) of \( Y \) whose inverse is, per force, linear as well. In the language of linear algebra, then, \( f \) is a linear isomorphism from \( X \) onto \( f(X) \), that is, \( X \) and \( f(X) \) are indistinguishable from the viewpoint of linear algebra as well.

Let us now return to the problem we posed above. Where \( X \) is a compact metrizable space, \( C(X) \) can be isometrically embedded in \( \ell_1 \); but can it be **linearly** isometrically embedded in \( \ell_1 \)? The answer is surprising: Yes! We will prove this fact now as an unexpected application of Stone-\( \beta \)-Cech compactification.

Let us begin with a simple, but very useful, auxiliary observation.

**Lemma 5.4.** \( C_b(X) \) and \( C(\beta X) \) are linearly isometrically isomorphic for any completely regular space \( X \).

**Proof.** Let \( X \) be a completely regular space so that \( \beta X \) exists. Then, for every \( f \) in \( C_b(X) \), there is a unique \( f^* \) in \( C(\beta X) \) such that \( f^*|_X = f \) (by definition of \( \beta X \)). We may then define the map \( \Phi : C_b(X) \to C(\beta X) \) by \( \Phi(f) := f^* \). This map is readily checked to be linear. Moreover, for any \( f \in C_b(X) \),

\[
\| f^* \|_\infty = \sup_{y \in \beta X} |f^*(y)| = \sup_{x \in X} |f^*(x)| = \sup_{x \in X} |f(y)| = \| f \|_\infty ,
\]

where the second equation follows from the fact that \( X \) is dense in \( \beta X \) and \( f^* \) is a continuous real map on \( \beta X \). Thus: \( \Phi \) is a linear isometry. As it is obvious that \( \Phi \) is surjective, we are done.

Recalling that \( \ell_\infty \) and \( C_b(\mathbb{N}) \) are the same spaces, the following is an immediate corollary of this lemma (which is actually what we need for our present purposes).
Corollary 5.5. $\ell_\infty$ and $C(\beta \mathbb{N})$ are linearly isometrically isomorphic.

Here is the second ingredient we need for our recipe.

Lemma 5.6. Let $X$ be a compact metric space. Then, $C(X)$ can be linearly isometrically embedded in $C(\beta \mathbb{N})$.

Proof. Since $X$ is a compact metric space, it is separable – see Exercise 2.11 of Chapter 7 – so there is a countable dense set $Z$ in $X$. Let us enumerate $Z$ as $\{z_1, z_2, \ldots\}$, and define $h : \mathbb{N} \to X$ by $h(i) := z_i$. By Proposition 4.1, there is a (unique) continuous map $H : \beta \mathbb{N} \to X$ such that $H(i) := z_i$ for each $i \in \mathbb{N}$. We then define $\Phi : C(X) \to C(\beta \mathbb{N})$ by $\Phi(f) := f \circ H$. This map is readily checked to be linear. Moreover, for any $f \in C(X)$,

$$\|\Phi(f)\|_\infty = \sup_{y \in \beta \mathbb{N}} |f(H(y))| = \sup_{i \in \mathbb{N}} |f(H(i))| = \sup_{z \in Z} |f(z)| = \|f\|_\infty,$$

where the second equation is true because $\mathbb{N}$ is dense in $\beta \mathbb{N}$ and $f \circ H$ is a continuous real map on $\beta \mathbb{N}$, while last equality holds because $Z$ is dense in $X$ and $f$ is continuous. Conclusion: $\Phi$ is a linear isometry from $C(X)$ into $C(\beta \mathbb{N})$.

Putting Corollary 5.5 and Lemma 5.6 together yields our claim:

Theorem 5.7. Let $X$ be a compact metric space. Then, $C(X)$ can be linearly isometrically embedded in $\ell_\infty$.

5.3 Digression: The Banach-Mazur Theorem

As startling as it is, Theorem 5.7 is actually nothing but the tip of the iceberg. With a little bit of help from elementary functional analysis, we can improve this result into something really special. All we need is the following fundamental theorem which was independently proved by Stefan Banach and Hans Hahn in 1927.

The Hahn-Banach Extension Theorem. Let $X$ be a normed linear space and $Z$ a linear subspace of $X$. Then, for every continuous linear map $f : Z \to \mathbb{R}$, there is a continuous linear map $F : X \to \mathbb{R}$ such that $F|_Z = f$ and $\|F\|_* = \sup_{z \in B_Z} |f(z)|$.

In words, this theorem says that every continuous linear map $f$ on a linear subspace of a normed linear space $X$ can be extended to such a map.

5 This section presumes familiarity with Section 3.2.3 of Chapter 7.
on the entire $X$ without increasing the operator norm of $f$. We will omit
the proof of this theorem here, as its argument (which is based on Zorn’s
Lemma) is not topological. You can find a proof for the Hahn-Banach
Extension Theorem in any introductory text on functional analysis.

In what follows, we only need the following corollary of the Hahn-Banach
Extension Theorem.

**Corollary 5.7.** Let $X$ be a normed linear space. Then,

$$
\|x\| = \sup_{f \in B_{X^*}} |f(x)| \quad \text{for every } x \in X.
$$

**Proof.** Take any $x$ in $X$. If $x = 0$, then $f(x) = 0$ for every $f \in X^*$, so our
claim is trivially true. Let us then assume that $\|x\| > 0$. It follows readily
from the definition of the operator norm that $|f(x)| \leq \|f\|^* \|x\| \leq \|x\|$ for
every $f \in B_{X^*}$, and hence, $\|x\| \geq \sup_{f \in B_{X^*}} |f(x)|$. To establish the converse
inequality, let $Z$ stand for $\{\lambda x : \lambda \in \mathbb{R}\}$ which is a (one-dimensional) linear
subspace of $X$. Define $g : Z \to \mathbb{R}$ by $g(\lambda x) := \|x\| \lambda$. Then, $g$ is a continuous
linear real map on $Z$, and $g(x) = \|x\|$. Moreover,

$$
\|g\|^* = \sup_{z \in B_z} |g(z)| = \sup \left\{ \|x\| |\lambda| : -\frac{1}{\|x\|} \leq \lambda \leq \frac{1}{\|x\|} \right\} = 1,
$$

so applying the Hahn-Banach Extension Theorem we find a a continuous
linear real map $F$ on $X$ such that $\|F\|^* = 1$ (so that $F \in B_{X^*}$) and $F(x) =
\|x\|$. It follows that $\|x\| \leq \sup_{f \in B_{X^*}} |f(x)|$, and we are done.

We use this to get:

**Proposition 5.8.** Every normed linear space $X$ can be linearly isometrically
embedded in $C(B_{X^*})$, where $B_{X^*}$ is endowed with the topology of pointwise
convergence.

**Proof.** For every $x$ in $X$, it is readily verified that the real map $f \mapsto f(x)$
on $B_{X^*}$ is continuous relative to the topology of pointwise convergence. We
may thus define the map $\Phi : X \to C(B_{X^*})$ by $\Phi(x)(f) := f(x)$, which
is readily checked to be linear. Moreover, by Corollary 5.7, $\|\Phi(x)\|_\infty =
\sup_{f \in B_{X^*}} |\Phi(x)(f)| = \sup_{f \in B_{X^*}} |f(x)| = \|x\|$ for every $x \in X$, so $\Phi$ is a
linear isometry.

In view of Alaoglu’s Theorem, therefore, we may conclude that every
normed linear space $X$ can be linearly isometrically embedded in $C(Y)$ for
some compact Hausdorff space $Y$. Further, suppose $X$ is separable. Then, by Proposition 2.9, $B_{X^*}$ is not only compact, but it is also metrizable. Combining this fact with Proposition 5.8, therefore, we get:

**The Banach-Mazur Theorem.** *Every separable normed linear space $X$ can be linearly isometrically embedded in $C(Y)$ for some compact metric space $Y$.***

*Warning.* This is actually a special case of the classical Banach-Mazur Theorem, although it is not really too far off the more general result. After all, with not too much effort, one can show that every compact metric space is the continuous image of the compact product space $\{0, 1\}^\infty$. This implies that $C(Y)$ is linearly isometrically isomorphic to a closed subset of $C(\{0, 1\}^\infty)$, for any compact metric space $Y$. Besides, it is not difficult to show that $C(\{0, 1\}^\infty)$ can be linearly isometrically embedded in $C[0, 1]$. Thus: *Every separable normed linear space $X$ can be linearly isometrically embedded in $C[0, 1]$. This is the classical Banach-Mazur Theorem.*

And now, we come to our tour de force. The Banach-Mazur Theorem says that a “copy” any given separable normed linear space is contained within $C(Y)$ for some compact metric space $Y$. But we have seen in Theorem 5.7 that, for any compact metric space $X$, a “copy” of $C(Y)$ can be found in $\ell_\infty$. Thus, we have the following amazing result:

**Theorem 5.9.** *Every separable normed linear space $X$ can be linearly isometrically embedded in $\ell_\infty$.***

This result is a very nice companion to the (far more elementary) Fréchet Embedding Theorem. The latter result says that every separable metric space is isometric to a subspace of $\ell_\infty$. In turn, Theorem 5.9 says that every separable normed linear space is linearly isometric to a subspace of $\ell_\infty$. 