Chapter 5
Products and Quotients

1 The Product Topology

A common theme in mathematics is to extend a mathematical structure that is defined on an arbitrary set to a structure that is defined on the Cartesian product of any given collection of sets in a consistent manner. Indeed, we have done this sort of an extension in Chapter 1 in the context of countably many metric spaces. By contrast, our task now is to describe how to turn the Cartesian product of an arbitrary collection of topological spaces into a topological space, and of course, we wish to do this in a consistent manner (in that the product of any number of singleton spaces and a topological space $X$ is homeomorphic to $X$). For expositional purposes, however, we will first look at the case of the product of finitely many spaces.

1.1 Finite Products of Topological Spaces

We have seen in Section 2.1 of the previous chapter that a subset of $\mathbb{R}^2$ is open iff it can be expressed as the union of a collection of open squares. More generally, for any positive integer $n$, we have $O \in \mathcal{O}_{\mathbb{R}^n}$ iff there are (possibly infinitely many) sets of the form $(a_1, b_1) \times \cdots \times (a_n, b_n)$ whose union equals $O$. In other words, the Euclidean topology of $\mathbb{R}^n$ is generated by the basis that consists of all sets of the form $O_1 \times \cdots \times O_n$ where each $O_i$ is an open interval. It is in this sense that we can think of $\mathbb{R}^n$ as the $n$-fold product of $\mathbb{R}$ as a topological space.

The Basis of the Product Topology

This way of looking at things generalizes in the natural way to make a new topological space out of any collection of finitely many topological spaces.
Definition. Let $n$ be an integer with $n \geq 2$, and $X_1, \ldots, X_n$ topological spaces. Put $X := X_1 \times \cdots \times X_n$. Then, it is readily checked that

$$B_X := \{O_1 \times \cdots \times O_n : O_1 \in \mathcal{O}_{X_1}, \ldots, O_n \in \mathcal{O}_{X_n}\},$$

is a basis for a topology on $X$. The topology generated by this basis is called the **product topology** on $X$. And, when we talk of the “product of the spaces $X_1, \ldots, X_n$,” what we mean is the topological space $(X, \mathcal{O}_X)$ where $\mathcal{O}_X$ is the product topology on $X$. (In this case, we refer to $X_i$ as the $i$th coordinate space of $X$.)

Unless stated otherwise explicitly, we shall always assume in this text that the Cartesian product of finitely many topological spaces is endowed with the product topology. This is a commonly adopted convention in the literature.

Remark. We can also use the bases of the coordinate spaces to describe the product topology. Indeed, where $B_{X_i}$ is a basis for the topology of the space $X_i$ for each $i = 1, \ldots, n$, a straightforward application of Proposition 2.1 of Chapter 4 shows that

$$\{B_1 \times \cdots \times B_n : B_1 \in B_{X_1}, \ldots, B_n \in B_{X_n}\}$$

is a basis for the product topology on $X$.

Remark. By definition, in the case of finitely many (coordinate) spaces $X_1, \ldots, X_n$, sets that are open in $X$ relative to the product topology are those that can be expressed as the union of the products of open sets in the coordinate spaces. Closed subsets of $X$ are similarly described. Indeed, using the notation of the definition above, we can say that $C_1 \times \cdots \times C_n$ is closed in $X$ relative to the product topology whenever $C_i$ is a closed subset of $X_i$ for each $i = 1, \ldots, n$, because

$$X \setminus (C_1 \times \cdots \times C_n) = ((X_1 \setminus C_1) \times X_2 \times \cdots \times X_n) \cap \cdots \cap (X_1 \times \cdots \times X_{n-1} \times (X_n \setminus C_n)).$$

Remark. Let $A_i$ be a subset of a topological space $X_i$ for each $i = 1, \ldots, n$. Then, by the previous remark, $\text{cl}(A_1) \times \cdots \times \text{cl}(A_n)$ is a closed set in $X$, and

\[\text{cl}(A_1) \times \cdots \times \text{cl}(A_n) = (O_1 \times \cdots \times O_n) \cap (U_1 \times \cdots \times U_n) = (O_1 \cap U_1) \times \cdots \times (O_n \cap U_n).\]
hence, it contains the closure of $A := A_1 \times \cdots \times A_n$ (relative to the product topology). Conversely, take any $x := (x_1, \ldots, x_n)$ in $\text{cl}(A_1) \times \cdots \times \text{cl}(A_n)$, and any $O \in \mathcal{O}_X(x)$. Then, for each $i$, there is an $O_i \in \mathcal{O}_{X_i}$ such that $x \in O_1 \times \cdots \times O_n \subseteq O$. Since $x_i \in \text{cl}(A_i)$, and $O_i$ is an open neighborhood of $x_i$ in $X_i$, we have $A_i \cap O_i \neq \emptyset$, for each $i = 1, \ldots, n$. Thus, $O_1 \times \cdots \times O_n$, and hence $O$, intersects $A$. In view of the arbitrary choice of $O$, therefore, we find that $x$ belongs to $\text{cl}(A_1 \times \cdots \times A_n)$. As $x$ was arbitrarily chosen in $\text{cl}(A_1) \times \cdots \times \text{cl}(A_n)$, then, we conclude:

$$\text{cl}(A_1 \times \cdots \times A_n) = \text{cl}(A_1) \times \cdots \times \text{cl}(A_n).$$

**Examples**

**Example 1.1.** Let $n \geq 2$ be an integer. Relative to the product topology on the $n$-fold product of $\mathbb{R}$, a subset $O$ of $\mathbb{R}^n$ is open iff for every $x$ in $O$ there are real numbers $\varepsilon_1, \ldots, \varepsilon_n > 0$ such that $(x_1 - \varepsilon_1, x_1 + \varepsilon_1) \times \cdots \times (x_n - \varepsilon_n, x_n + \varepsilon_n)$ is contained in $O$. It follows that the product topology on $\mathbb{R}^n$ is none other than the metric topology of $\mathbb{R}^{n,1}$. But we have seen in Example 1.10 of Chapter 4 that the metric topologies of $\mathbb{R}^{n,1}$ and $\mathbb{R}^{n,p}$ are the same for any $p \in [1, \infty]$, so there is no difference between these spaces from the vantage of topology. Thus, the product topology on $\mathbb{R}^n$ is the same as the metric topology of $\mathbb{R}^{n,p}$ for any $p \in [1, \infty]$. In particular, the standard topology of $\mathbb{R}^n$ is the product topology on the $n$-fold product of $\mathbb{R}$.

By definition, the product topology on $\mathbb{R}^n$ is the metric topology of $\mathbb{R}^{n,1}$. Another way of saying this is that the topology induced by the product metric on $\mathbb{R}^n$ is in agreement with the product topology on $\mathbb{R}^n$. This is a special case of a general situation.

**Example 1.2.** For any integer $n \geq 2$, take $n$ many metric spaces $(X_1, d_1), \ldots, (X_n, d_n)$, and put $X := X_1 \times \cdots \times X_n$. Then, the metric topology on $X$ induced by the product metric $\rho$ is the same as the product topology on $X$, where we view each $X_i$ as a topological space relative to its metric topology. Indeed, the set of all $B_{X_i}(x_i, \varepsilon_i)$, where $x_i$ varies over $X_i$ and $\varepsilon_i > 0$, is a basis for the metric topology of $X_i$, for each $i$. As we have remarked above, this means that the collection $\mathcal{B}$ of all $B_{X_1}(x_1, \varepsilon_1) \times \cdots \times B_{X_n}(x_n, \varepsilon_n)$, where $x_i \in X_i$ and $\varepsilon_i > 0$ for each $i$, is a basis for the product topology on $X$. But, by Example 4.8 of Chapter 1, this collection is also a basis for the metric topology of $X$. Thus: $\mathcal{O}_{(X, \rho)} = \mathcal{O}_X$. 

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We have seen in Example 1.10 of Chapter 2 that the projection maps on the product of finitely many metric spaces are continuous. This fact extends to the present context.

**Example 1.3.** For any integer \( n \geq 2 \), take \( n \) many topological spaces \( X_1, \ldots, X_n \), and let \( X \) be the product of these spaces. Then, for any \( k \in \{1, \ldots, n\} \), the \( k \)th projection map \( \text{proj}_k : X \to X_k \), which maps \( x := (x_1, \ldots, x_n) \) to \( x_k \), is continuous. Indeed, if \( O_1 \in \mathcal{O}_{X_1} \), then \( \text{proj}_1^{-1}(O_1) = O_1 \times X_2 \times \cdots \times X_n \), which means that \( \text{proj}_1^{-1}(O_1) \) is open in \( X \), being an element of the basis for the product topology on \( X \). Exactly the same argument applies to all the other \( k \)s as well.

**Remark.** In the setting of Example 1.3, the image of any element \( O_1 \times \cdots \times O_n \) of the basis \( \mathcal{B}_X \) for the product topology on \( X \) is \( O_k \) under the map \( \text{proj}_k \), \( k = 1, \ldots, n \). It follows that all of the projection maps on \( X \) are open.

**Warning.** A projection map is always open, but it need not be closed. For instance, \( C := \{ x \in \mathbb{R}^2 : \ln x_1 + \ln x_2 = 0 \} \) is a closed subset of \( \mathbb{R} \times \mathbb{R} \) – why? – but \( \text{proj}_1(C) = \mathbb{R} \setminus \{0 \} \).

**The Subbasis of the Product Topology**

Example 1.3 actually tells only one part of the story. Indeed, in the setting of that example, we have

\[
O_1 \times \cdots \times O_n = \bigcap_{i=1}^{n} \text{proj}_i^{-1}(O_i)
\]

for every \( O_i \in \mathcal{O}_{X_i}, i = 1, \ldots, n \). In other words, taking the intersections of the members of the collection

\[
\mathcal{S} := \{ \text{proj}_i^{-1}(O_i) : O_i \in \mathcal{O}_{X_i}, i = 1, \ldots, n \},
\]

we obtain exactly the basis \( \mathcal{B}_X \) of the product topology on \( X \). Put differently still:

**Proposition 1.1.** For any integer \( n \geq 2 \), let \( X_1, \ldots, X_n \) be topological spaces. Then, the product topology on \( X := X_1 \times \cdots \times X_n \) is precisely the topology generated on \( X \) by using the collection \( \mathcal{S} \) in (1) as a subbasis.

We can still say a bit more. We know that the product topology on \( X \) renders all \( (n \text{ of}) \) the projection maps continuous. Now notice that if
\(O\) is another such topology on \(X\), then, for any \(i = 1, \ldots, n\), this topology must contain any set of the form \(\text{proj}_i^{-1}(O_i)\), where \(O_i \in \mathcal{O}_{X_i}\). It follows that this topology contains \(S\) in (1), and hence, the entire basis of the product topology on \(X\). In other words, this topology contains the product topology. We conclude: The product topology is the smallest topology on \(X\) that renders each projection map on \(X\) continuous. (Recalling Exercise 4.16 of Chapter 4, we can restate this fact as: The product topology is the weak topology on \(X\) induced by the set of all projection maps on \(X\).)

**Continuity of Product-Valued Functions**

**Example 1.4.** For any integer \(n \geq 2\), take any topological spaces \(Y_1, \ldots, Y_n\), and let \(Y\) stand for the product of these spaces. Now let \(X\) be any topological space, and for each \(i\), take any \(f_i : X \to Y_i\). Then, the map \(f : X \to Y\), defined by

\[
f(x) := (f_1(x), \ldots, f_n(x)),
\]

is continuous iff each (component map) \(f_i\) is continuous. The “only if” part of this claim is true because \(\text{proj}_i : Y \to Y_i\) is continuous, and \(f_i = \text{proj}_i \circ f\), for each \(i = 1, \ldots, n\). Conversely, suppose each \(f_i\) is continuous. Fix any \(k\) in \(\{1, \ldots, n\}\), and consider any subset of \(Y\) of the form \(\text{proj}_k^{-1}(O_k)\) where \(O_k\) is an open set in \(Y_k\). Clearly,

\[
f^{-1}(\text{proj}_k^{-1}(O_k)) = (\text{proj}_k \circ f)^{-1}(O_k) = f_k^{-1}(O_k),
\]

and hence, since \(f_k\) is continuous, we find that \(f^{-1}(\text{proj}_k^{-1}(O_k))\) is open in \(X\). Conclusion: \(f^{-1}(S)\) is open in \(X\) for every element \(S\) of a subbasis that generates the product topology on \(Y\). As we have remarked right after Proposition 4.1 of Chapter 4, this is equivalent to say that \(f\) is continuous.

We can deduce many other results from this fact. For instance, suppose that \(X\) is itself the product of \(n\) many topological spaces, say, \(X_1, \ldots, X_n\). Then, for any \(g_i \in \mathcal{C}(X_i, Y_i)\), \(i = 1, \ldots, n\), the map \(g : X \to Y\), defined by

\[
g(x_1, \ldots, x_n) := (g_1(x_1), \ldots, g_n(x_n)),
\]

is continuous. To see this, denote the \(i\)th projection map from \(X\) onto \(X_i\) by \(p_i\), and note that \(h_i := g_i \circ p_i\) is a continuous real map on \(X\) (for each \(i\)). Since \(g(x) = (h_1(x), \ldots, h_n(x))\) for every \(x \in X\), therefore, applying what we have found in the previous paragraph yields our claim.

**Remark.** Here is a nice application of the previous observation. Suppose, in the context of the previous example, we have \(X_1 \cong Y_1, \ldots, X_n \cong Y_n\). Then,
for each \( i = 1, \ldots, n \), there is an homeomorphism \( f_i \) from \( X_i \) onto \( Y_i \). But then the map \( (x_1, \ldots, x_n) \mapsto (f_1(x_1), \ldots, f_n(x_n)) \) is an homeomorphism from \( X \) onto \( Y \). Thus: If \( X_i \cong Y_i \) for each \( i = 1, \ldots, n \), then \( X \cong Y \). (Exercise: Converse of this is also true. Why?)

**Example 1.5.** Please look back at Example 1.12 of Chapter 2. Our findings in Example 1.4 show that everything we said in that example remains valid when \( X, X_1, \ldots, X_n \) are topological spaces as well. In particular, \( \lambda_1 f_1 + \cdots + \lambda_n f_n \), as well as \( \prod^n_{i=1} f_i \), is a continuous real map on \( X \), where \( X \) is a topological space, \( f_1, \ldots, f_n \in C(X) \), and \( \lambda_1, \ldots, \lambda_n \in \mathbb{R} \).

**Example 1.6.** Let \( X \) be a topological space, and take any \( f, g \in C(X) \). We claim that \( \min\{f, g\} \), that is, the real map on \( X \) which maps \( x \) to \( \min\{f(x), g(x)\} \), is continuous. Indeed, it is an easy exercise in calculus to check that \( (a, b) \mapsto \min\{a, b\} \) is a continuous real map on \( \mathbb{R}^2 \). But \( \min\{f, g\} \) is none other than the composition of this minimum map with the \( \mathbb{R}^2 \)-valued map \( x \mapsto (f(x), g(x)) \) on \( X \). As we already know that the latter map is continuous, therefore, \( \min\{f, g\} \) is continuous.\(^2\)

By applying the composition operator inductively we can extend this observation to the case of any finitely many continuous maps. That is, if \( f_1, \ldots, f_n \in C(X) \), then \( \min\{f_1, \ldots, f_n\} \in C(X) \). Of course, we also have \( \max\{f_1, \ldots, f_n\} \in C(X) \). We can either prove this by the analogous reasoning, or deduce it readily from what we have just found and the fact that \( \max\{f_1, \ldots, f_n\} = -\min\{-f_1, \ldots, -f_n\} \).

**A Retake on the Hausdorff Property**

For any nonempty set \( X \), by the **diagonal** of \( X \times X \), we mean the following subset of \( X \times X \):

\[
\Delta_X := \{ (x, x) : x \in X \}.
\]

When \( X \) is a topological space, it is of interest to know whether or not this set is closed in \( X \times X \) relative to the product topology. Indeed, the closedness of the diagonal of \( X \times X \) characterizes “Hausdorffness” of \( X \), which is a surprisingly useful fact.

\(^2\)We could also give a more direct proof. Put \( A := \{ x \in X : f(x) \geq g(x) \} \) and \( B := \{ x \in X : f(x) \leq g(x) \} \). Notice that \( A \) is the inverse image of the closed set \([0, \infty)\) under \( f - g \), and \( B \) is that under \( g - f \). As continuity of \( f \) and \( g \) entail that of \( f - g \) and \( g - f \), therefore, \( A \) and \( B \) are closed subsets of \( X \), and of course, \( f(x) = g(x) \) for every \( x \in A \cap B \). By the Pasting Lemma, then, the real map on \( X \) which equals \( g \) on \( A \) and \( f \) on \( B \) must be continuous. But this map is none other than \( \min\{f, g\} \).
Proposition 1.2. A topological space $X$ is Hausdorff if, and only if, $\Delta_X$ is closed in $X \times X$.

Proof. Let us write $V$ for $(X \times X) \setminus \Delta_X$; we wish to show that $X$ is Hausdorff iff $V$ is open in $X \times X$. Assume first that $X$ is Hausdorff, and take any $(x, y)$ in $V$. Then, $x \neq y$, so there are disjoint open sets $O$ and $U$ in $X$ such that $x \in O$ and $y \in U$. But then, $O \times U$ is an open neighborhood of $(x, y)$ in $X \times X$ that lies entirely outside $\Delta_X$ (because $O$ and $U$ are disjoint). Given the arbitrary choice of $(x, y)$, then, we conclude that $V$ is open in $X \times X$. Conversely, assume that $V$ is open in $X \times X$, and take any distinct $x$ and $y$ in $X$. Then, $(x, y) \in V$, so by definition of the product topology, we can find open subsets $O$ and $U$ of $X$ such that $(x, y) \in O \times U \subseteq V$. But then $O \times U$ does not intersect $\Delta_X$, which means that $O$ and $U$ are disjoint. Given the arbitrary choice of $x$ and $y$, we conclude that $X$ is Hausdorff.

Here is an application of this result.

Example 1.7. Let $X$ and $Y$ be topological spaces and $f \in C(X, Y)$. We define the graph of $f$ as the following subset of $X \times Y$:

$$\text{graph}(f) := \{(x, f(x)) : x \in X\}.$$ 

If $Y$ is Hausdorff, then this set must be closed. That is: Any continuous map from a topological space into a Hausdorff space has a closed graph. Why? Well, one (slick!) way of seeing this is to define the map $F : X \times Y \to Y \times Y$ by $F(x, y) := (f(x), y)$, and to notice that $\text{graph}(f) = F^{-1}(\Delta_Y)$. Since $F$ is continuous (Example 1.4), and $\Delta_Y$ is closed (Proposition 1.2), therefore, $\text{graph}(f)$ is closed.\(^3\)

1.2 Product Representation of Special Spaces

The description of some topological spaces of geometric interest are simplified when we express them as products of finitely many (simpler) spaces. As a trivial example, take any positive integer $n$, and let $\mathbb{R}^{n, \infty}$ stand for the space $\mathbb{R}^n$ with the metric topology induced by the sup-norm. Now let $B$ stand for the closed unit ball of this space, that is, $B := \{x \in \mathbb{R}^n : \|x\|_\infty \leq 1\}$. As a set, this is none other than the $n$-fold Cartesian product of the interval $[-1, 1]$. This is also the case from the topological viewpoint. True, as a subspace of $\mathbb{R}^{n, \infty}$, the topology on $B$ is the metric topology induced

\(^3\)Alternatively, one can use Proposition 5.4 of Chapter 4 to give a more direct proof of this fact.
by the sup-norm. But we know that $d_1$ and $d_\infty$ are equivalent metrics on $\mathbb{R}^n$ (Example 6.3 of Chapter 1), so the identity map on $\mathbb{R}^n$ is an homeomorphism from $\mathbb{R}^{n,1}$ onto $\mathbb{R}^{n,\infty}$. Thus, $B$ is homeomorphic to $[-1,1]^n$ where the latter is endowed with the $d_1$ metric. It then follows from Example 1.2 that $B$ is homeomorphic to the topological product of $n$ copies of the interval $[-1,1]$. In other words, topologically speaking, $B$ "is" $[-1,1] \times \cdots \times [-1,1]$ with the product topology.

This is overanalyzing a trivial situation, to be sure. Less trivial illustrations follow.

**Example 1.8.** *(The Cylinder)* Consider the (hollow) cylinder in $\mathbb{R}^3$ defined as

$$C := \{ x \in \mathbb{R}^3 : x_1^2 + x_2^2 = 1 \text{ and } 0 \leq x_3 \leq 1 \}.$$ 

Then, $C \cong S^1 \times [0,1]$. Indeed, it is readily checked that the identity map $x \mapsto x$ is an homeomorphism from $C$ onto $S^1 \times [0,1]$.

**Remark.** This example is a (very) special case of a general situation. Let $X$ and $Y$ be two topological spaces, and $A$ and $B$ nonempty subsets of $X$ and $Y$, respectively. There are two natural topologies we can define on $A \times B$.

First, we can view $A \times B$ as a subspace of $X \times Y$. This endows $A \times B$ with a topology that is generated by a basis that consists of sets of the form

$$(O \times U) \cap (A \times B) \text{ where } O \in \mathcal{O}_X \text{ and } U \in \mathcal{O}_Y.$$ 

(In Example 1.8, where $X$ is $\mathbb{R}^2$ and $Y$ is $\mathbb{R}$, this is the topology on $C$.) Second, we can view $A$ as a subspace of $X$, and $B$ of $Y$, and then impose the product topology on $A \times B$. This endows $A \times B$ with a topology that is generated by a basis that consists of sets of the form

$$(O \cap A) \times (U \cap B) \text{ where } O \in \mathcal{O}_X \text{ and } U \in \mathcal{O}_Y.$$ 

(In Example 1.8, this is the topology that $S^1 \times [0,1]$ carries.) Fortunately, these topologies are the same (simply because $(O \times U) \cap (A \times B) = (O \cap A) \times (U \cap B)$ holds true in general), so these two natural approaches are fully compatible with each other.

**Example 1.9.** *(The Annulus)* The annulus in $\mathbb{R}^2$ is defined as

$$A := \{ x \in \mathbb{R}^2 : 1 \leq \|x\|_2 \leq 2 \}.$$
It is fairly easy to see that this subspace of $\mathbb{R}^2$ is homeomorphic to the cylinder $C$ of Example 1.8. Indeed, routine computations show that $f : C \to A$, defined by

$$f(x_1, x_2, x_3) := ((1 + x_3)x_1, (1 + x_3)x_2),$$

is a bijection, whose inverse is the map

$$(x_1, x_2) \mapsto \left(\frac{x_1}{\|x\|_2}, \frac{x_2}{\|x\|_2}, \|x\|_2 - 1\right)$$

from $A$ onto $C$. Moreover, in view of Example 1.4, it is easy to check that both of these maps are continuous, and hence, $f$ is an homeomorphism.

And now combining this observation with the previous example, we find that, topologically speaking, the annulus $A$ “is” the product of the unit circle with the unit interval, that is, $A \cong S^1 \times [0, 1]$. (Please note that this is not trivial. $A$ lies in $\mathbb{R}^2$ while $S^1 \times [0, 1]$ in $\mathbb{R}^3$. Put precisely, what we do here is embedding $S^1 \times [0, 1]$ in $\mathbb{R}^2$.)

**Example 1.10.** (The 2-Dimensional Torus) Consider the surface of revolution obtained by rotating the unit circle in the $xz$-plane (that is, in $\mathbb{R} \times \{0\} \times \mathbb{R}$) centered at $(2, 0, 0)$ about the $z$-axis (that is, about $\{0\} \times \{0\} \times \mathbb{R}$), the plane containing the circle being always perpendicular to the $xy$-plane (that is, to $\mathbb{R} \times \mathbb{R} \times \{0\}$). This set, which is called the (2-dimensional) torus, can be described explicitly as

$$T^2 := \left\{ x \in \mathbb{R}^3 : \left(\sqrt{x_1^2 + x_2^2} - 2\right)^2 + x_3^2 = 1 \right\},$$

but this is by no means the most efficient way of working with this set. Instead, consider the circles

$$C_1 := \{(a, b, 1) : a^2 + b^2 = 4\} \quad \text{and} \quad C_2 := \{(c, 0, d) : (c - 2)^2 + d^2 = 1\}.$$

Let us map $C_1 \times C_2$ into $T^2$ by mapping $((a, b, 1), (c, 0, d))$ to the 3-vector to which $(c, 0, d)$ is carried when $C_2$ is rotated about the $z$-axis until it touches $(a, b, 1)$. One can show that this map $f : C_1 \times C_2 \to T^2$ can be described more explicitly as

$$f((a, b, 1), (c, 0, d)) := \left(\frac{ac}{2}, \frac{bc}{2}, d\right).$$
It is readily checked that $f$ is a bijection, with $g : T^2 \to C_1 \times C_2$, defined by

$$g(x_1, x_2, x_3) := \left( \left( \frac{2x_1}{\sqrt{x_1^2 + x_2^2}}, \frac{2x_2}{\sqrt{x_1^2 + x_2^2}}, 1 \right), \left( \sqrt{x_1^2 + x_2^2}, 0, x_3 \right) \right),$$

being its inverse. Moreover, it is fairly easy to check that both of these maps are continuous.\(^4\) Thus: $f$ is a homeomorphism.

We have just shown that $C_1 \times C_2$ and $T^2$ are homeomorphic. But, from the topological perspective, there is no difference between either of these circles and the unit circle, that is, $C_i \cong S^1$ for each $i = 1, 2$. (Why?)

Consequently, recalling our Remark after Example 1.4, we may conclude that

$$T^2 \cong S^1 \times S^1.$$ 

Thus, everything topological there is to learn about the (2-dimensional) torus can be learned from the simple product space $S^1 \times S^1$. In fact, for this reason, many texts on topology “define” the torus as $S^1 \times S^1$. Such texts view the map $f$ above as the embedding of $S^1 \times S^1$ (or more precisely, $C_1 \times C_2$) into $\mathbb{R}^3$. (Exercise 1.7 below takes this way of looking at things a bit further.)

### 1.3 Arbitrary Products of Topological Spaces

Our task now is to find a way of taking the “product” of a given collection of topological spaces when this collection need not be finite, or even countable. Let us first understand how we can think about this matter from a purely set-theoretic perspective.

Let $I$ be a nonempty (index) set, and let $X_i$ be a set for each $i \in I$. By the **Cartesian product** of the collection $\{X_i : i \in I\}$, we mean the collection of all maps $x : I \to \bigcup_{i \in I} X_i$ such that $x(i) \in X_i$ for each $i \in I$. We may denote this collection as

$$X_{i \in I} X_i \quad \text{or} \quad X\{X_i : i \in I\}. $$

(Notice that if $I$ is $\{1, \ldots, n\}$ or $\mathbb{N}$, this definition reduces to how we usually think of the Cartesian product of countably many sets.)

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\(^4\)It is plain that $(x_1, \ldots, x_6) \mapsto (x_1x_4/2, x_2x_4/2, x_6)$ is a continuous map from $\mathbb{R}^6$ to $\mathbb{R}^3$. (Recall Example 1.12 of Chapter 2.) Now consider $C_1 \times C_2$ as a subspace of $\mathbb{R}^6$ – the Remark preceding Example 1.9 says that we can do this – and notice that restricting this map to $C_1 \times C_2$ yields precisely the map $f$. Thus, by Proposition 4.5 of Chapter 4, $f$ is continuous. I leave it to you to show that $g$ is continuous as well.
Let us write $X$ for $X_i \in I X_i$. For each $i$ in $I$, the **projection map onto** $X_i$ is the function $\text{proj}_i : X \rightarrow X_i$ defined by $\text{proj}_i(x) := x(i)$. As usual, for any $i \in I$ and $S \subseteq X_i$, we denote the set $\{ x \in X : \text{proj}_i(x) \in S \}$, the inverse image of $S$ under $\text{proj}_i$, by $\text{proj}_i^{-1}(S)$; thus

$$x \in \text{proj}_i^{-1}(S) \quad \text{iff} \quad x(j) \in X_j \text{ for every } j \in I \setminus \{i\} \text{ and } x(i) \in S.$$  

If, for instance, $I = \mathbb{N}$, then $\text{proj}_1^{-1}(S)$ equals $S \times X_2 \times X_3 \times \cdots$ for any subset $S$ of $X_1$, $\text{proj}_2^{-1}(T)$ equals $X_1 \times T \times X_3 \times \cdots$ for any subset $T$ of $X_2$, and so on.

Suppose now that $X_i$ is a topological space for each $i \in I$. We would like to define a topology on $X$ which would reduce to the product topology we defined in Section 1.1 when $I$ is finite. An immediate idea is to mimic what we have done earlier. Indeed, $\{X_i \in \mathcal{O}_i : \mathcal{O}_i \in \mathcal{O}_{X_i} \text{ for each } i \in I \}$ is a basis for a topology on $X$, and it seems like a good idea to define as the “product topology” on $X$ the topology generated by this basis. This topology is actually called the **box topology** on $X$, and while it is in a sense a natural topology, it does suffer from some serious shortcomings. (We will explore some of these in various exercises throughout the text. See, for instance, Exercise 1.2 below.)

Instead, we would like to use the characterization we obtained in Proposition 1.1 as the starting point of our definition. Consequently, we define

$$S_i := \{ \text{proj}_i^{-1}(O_i) : O_i \in \mathcal{O}_{X_i} \} \text{ for each } i \in I,$$

and let

$$\mathcal{S} := \bigcup \{ S_i : i \in I \}.$$  

(In other words, any $S$ in $\mathcal{S}$ is the Cartesian product of open sets $O_i$ for each $i \in I$ such that $O_i \neq X_i$ for at most one $i$.) The **product topology** on $X$ is defined as the topology generated by using $\mathcal{S}$ as a subbasis. When it is endowed with this topology, we refer to $X$ as the **topological product** of $\{ X_i : i \in I \}$. From now on, when we talk about the “product of $\{ X_i : i \in I \}$,” we always mean the topological space $(X, \mathcal{O}_X)$ where $\mathcal{O}_X$ is the product topology on $X$.

Putting together all intersections of finitely many elements of $\mathcal{S}$ yields the following basis for the product topology on $X$:

$$B_X := \{ X_i \in \mathcal{O}_i : O_i \in \mathcal{O}_{X_i} \text{ for each } i \text{ and } O_i \neq X_i \text{ for finitely many } i \}.$$  

Thus, a typical element of this basis looks like $X_i \in \mathcal{O}_i$ where $O_i$ is an open subset of $X_i$ for each $i \in I$, and $O_i = X_i$ for all but finitely many $i \in I$. (The
italicized part is where the product topology differs from the box topology. Clearly, the product topology is coarser than the box topology.)

**Example 1.11.** $(0, 1)^\infty$ is an open subset of $\mathbb{R}^\infty$ relative to the box topology; this set is in fact one of the elements of the basis that generates this topology. Yet, $(0, 1)^\infty$ is not open in $\mathbb{R}^\infty$ relative to the product topology. After all, any basis element for the product topology on $\mathbb{R}^\infty$ is of the form $O_1 \times O_2 \times \cdots$ with only finitely many of the $O_i$s being distinct from $\mathbb{R}$. Thus, no basis element for the product topology on $\mathbb{R}^\infty$ may possibly be contained in $(0, 1)^\infty$.

Our general definition of the product topology conforms fully with the definition we gave in the case of the product of finitely many topological spaces. In particular, the product and box topologies coincide in the latter case; this is what Proposition 1.1 says. But, as the previous example demonstrates, these topologies are in general distinct in the case of infinite products.

**Remark.** As in the finite case, we can also use the bases of the coordinate spaces to describe the product topology. Indeed, using the notation above, suppose $B_{X_i}$ is a basis for the topology of $X_i$ for each $i \in I$. Then, a straightforward application of Proposition 2.1 of Chapter 4 shows that

$$\{X_i \in B_i : B_i \in B_{X_i} \cup \{X_i\} \text{ for each } i \text{ and } B_i \neq X_i \text{ for finitely many } i\}$$

is a basis for the product topology on $X$.

**Remark.** Our construction of the product topology ensures that the projection map from the topological product of $\{X_i : i \in I\}$ onto $X_k$ is continuous (for any $k \in I$). In fact, just as in the finite case, a closer inspection of this construction shows that the product topology on $X := \prod_{i \in I} X_i$ is the smallest topology on this set that renders each projection map on $X$ continuous. (In other words, where $\text{proj}_i$ stands for the projection map from $X$ onto $X_i$, the product topology on $X$ is the weak topology induced by $\{\text{proj}_i : i \in I\}$; recall Exercise 4.16 of Chapter 4.)

Example 1.4 extends beautifully to the present setup.

**Example 1.12.** For any nonempty (index) set $I$, let $Y_i$ be a topological space for each $i \in I$, and let $Y$ stand for the topological product of $\{Y_i : i \in I\}$. In turn, let $X$ be any topological space, and for each $i \in I$, take any $f_i : X \to Y_i$. Now define the map $f : X \to Y$ by

$$f(x)(i) := f_i(x) \quad \text{for each } i \in I.$$
(Recall that an element of $Y$, and hence any $f(x)$, is a function from $I$ into $\bigcup_{i \in I} Y_i$.) Then, $f$ is continuous iff $f_i$ is continuous for each $i \in I$. The proof is identical to the one we gave in Example 1.4.\footnote{This is one place where we see an important difference between the product and box topologies. Indeed, as you will be asked to demonstrate in the context of a specific example below, the “if” part of what we have just found would fail if we used the box topology on $Y$.}

The product topology is a rather friendly topology in that most properties that are satisfied by a given collection of topological spaces is also satisfied by the product of these spaces. In particular, the following result shows that the separation-by-open-set properties of a collection of topological spaces are transmitted to the topological product of these spaces.

**Proposition 1.3.** Let $I$ be a nonempty set, $X_i$ a topological space for each $i \in I$, and $X$ the topological product of $\{X_i : i \in I\}$. Then, $X$ is a $T_0$- (or $T_1$, or Hausdorff) space if, and only if, $X_i$ is a $T_0$- (or $T_1$, or Hausdorff) space for each $i \in I$.

**Proof.** We prove this for the case of Hausdorff spaces, but the same argument applies to the other two cases as well. Suppose each $X_i$ is a Hausdorff space, and take any two distinct points $x$ and $y$ in $X$. Then, $x(j) \neq y(j)$ for some $j \in I$, so there are disjoint open subsets $O_j$ and $U_j$ of $X_j$ such that $x(j) \in O_j$ and $y(j) \in U_j$. Then, $\text{proj}^{-1}_j(O_j)$ and $\text{proj}^{-1}_j(U_j)$ are disjoint open subsets of $X$ which contains $x$ and $y$, respectively. Conclusion: $X$ is Hausdorff. Conversely, suppose $X$ is a Hausdorff space, and fix an arbitrary $j$ in $I$. Let $z_j$ and $w_j$ be two distinct points in $X_j$. Now take any $u \in X$, and define the points $x$ and $y$ in $X$ as

$$x(i) := \begin{cases} z_j, & \text{if } i = j, \\ u(i), & \text{otherwise,} \end{cases} \quad \text{and} \quad y(i) := \begin{cases} w_j, & \text{if } i = j, \\ u(i), & \text{otherwise.} \end{cases}$$

Then, $x$ and $y$ are distinct points in $X$, so there are disjoint open subsets $O$ and $U$ of $X$ such that $x \in O$ and $y \in U$. By definition of the product topology, then, $x$ is contained in $\bigcap_{i \in I} O_i$, and $y$ in $\bigcap_{i \in I} U_i$, where $O_i \in \mathcal{O}_X$ for each $i$, and $O_i = X_i$ for all but finitely many $i$, and similarly for $\{U_i : i \in I\}$. Obviously, $z_j \in O_j$ and $w_j \in U_j$. Furthermore, $u(i) \in O_i \cap U_i$ for every $i \in I \setminus \{j\}$. Since $X_{i \in I} O_i$ and $X_{i \in I} U_i$ are disjoint, therefore, we must have $O_j \cap U_j = \emptyset$. Conclusion: $X_j$ is a Hausdorff space.

Things are also pleasant in the case of countable-basis and metrizability properties for the topological product of countably many spaces, and even
more pleasant in terms of separability. But we will look into these matters in
the next section. For now, we only note that the products of homeomorphic
topological spaces are themselves homeomorphic.

**Proposition 1.4.** Let \( I \) be a nonempty set, and let \( X_i \) and \( Y_i \) be homeo-
morphic topological spaces for each \( i \in I \). Then, the topological products
of \( \{X_i : i \in I\} \) and \( \{Y_i : i \in I\} \) are homeomorphic.

**Proof.** For each \( i \in I \), pick any homeomorphism \( f_i \) from \( X_i \) onto \( Y_i \).
Let us write \( X \) for the topological product of \( \{X_i : i \in I\} \), and \( Y \) for that
of \( \{Y_i : i \in I\} \). We define \( f : X \to Y \) by \( f(x)(i) := f_i(x(i)) \) for every \( i \in I \).
It is plain that \( f \) is a bijection. Now let us denote the projection map from
\( X \) onto \( X_i \) by \( p_i \), and that from \( Y \) into \( Y_i \) by \( q_i \). Then, where \( g_i := f_i \circ p_i \)
and \( h_i := f_i^{-1} \circ q_i \), we have

\[
  f(x)(i) = g_i(x) \quad \text{and} \quad f^{-1}(y)(i) = h_i(y)
\]

for each \( i \in I \), and each \( x \in X \) and \( y \in Y \). As both \( p_i \) and \( q_i \) are continuous,
both \( g_i \) and \( h_i \) are continuous, for each \( i \in I \). Hence, by what we have found
in Example 1.12, both \( f \) and \( f^{-1} \) are continuous.

### 1.4 Convergence in Product Spaces

We have seen in Section 5.2 of Chapter 1 that convergence of a sequence in
the product of countably many metric spaces is coordinatewise. That is, such
a sequence converges to a point in that space if the corresponding sequence
of the \( i \)th terms of that sequence converges to the \( i \)th term of the limit (in
the \( i \)th metric space) for each \( i \). This fact extends to our present setup. That
is, “convergence” of a sequence in the product of an arbitrary collection of
topological spaces is equivalent to “coordinatewise convergence” as well. In
fact, we may now state this result in a more general fashion by using nets
instead of sequences.

**Proposition 1.5.** Let \( I \) be a nonempty set, \( X_i \) a topological space for each
\( i \in I \), and \( X \) the topological product of \( \{X_i : i \in I\} \). Take any net \( (x_\alpha) \)
in \( X \) and any \( x \in X \). Then, \( x_\alpha \to x \) if, and only if, \( x_\alpha(i) \to x(i) \) for each
\( i \in I \).

**Proof.** Suppose \( x_\alpha \to x \). For any \( i \in I \) and any open neighborhood
\( O_i \) of \( x(i) \), \( \proj_i^{-1}(O_i) \) is an open neighborhood of \( x \). If \( x_\alpha \to x \), therefore,
\( (x_\alpha) \) is eventually in \( \proj_i^{-1}(O_i) \), and hence \( (x_\alpha(i)) \) is eventually in \( O_i \), for
each \( i \in I \). This proves the “only if” part of our proposition. Conversely,
suppose \( x_\alpha(i) \to x(i) \) for each \( i \in I \). It readily follows from this that \( (x_\alpha) \) is eventually in every \( S \in \mathcal{S} \) that contains \( x \), where \( \mathcal{S} \) is the subbasis for the product topology on \( x \) (as specified above). As the index poset of \((x_\alpha)\) is directed, \( (x_\alpha) \) is eventually in \( \bigcap T \) for every nonempty finite subset \( T \) of \( \mathcal{S} \) as well. Put differently, \((x_\alpha)\) is eventually in \( B \) for every \( B \in \mathcal{B} \), where \( \mathcal{B} \) is a basis for the product topology on \( X \). It follows that \( x_\alpha \to x \).

In words, convergence of a net in the topological product of a collection of topological spaces is equivalent to convergence of each of the nets that are made out of the “ith” terms of that net. It is in this sense that coordinatewise. Given that we view any one member of a product space as a function (on an index set), we may also state this fact loosely as “convergence with respect to the product topology is pointwise convergence.” This is what Proposition 1.5 says, thereby motivating the following terminology which is commonly used in the theory of function spaces.

**Example 1.13.** Let \( X \) be a nonempty set and \( Y \) a topological space. Recall that \( Y^X \) stands for the collection of all maps from \( X \) into \( Y \). Here is a standard way of topologizing \( Y^X \). For any \( x \) in \( X \) and any open subset \( O \) of \( Y \), define

\[
U(x, O) := \{ f \in Y^X : f(x) \in O \},
\]

and let \( \mathcal{S} := \{ U(x, O) : x \in X \text{ and } O \in \mathcal{O}_Y \} \). The topology on \( Y^X \) generated by using \( \mathcal{S} \) as a subbasis is called the **topology of pointwise convergence** on \( Y^X \). The name for this topology owes to the following fact: A net \( (f_\alpha) \) in \( Y^X \) converges to a map \( f : X \to Y \) under the topology of pointwise convergence if \( f_\alpha(x) \to f(x) \) for each \( x \in X \) (relative to the topology of \( Y \)). Indeed, the topology of pointwise convergence on \( Y^X \) is none other than the topology that makes \( Y^X \) the topological product of \( \{X_i : i \in I\} \) where \( I = X \) and \( X_i = Y \) for each \( i \in I \). Thus, our claim follows readily from Proposition 1.5.

**Warning.** From the vantage of set theory, there is no difference between \( \mathbb{R}^n \) and \( \mathbb{R}^{\{1,\ldots,n\}} \) for any positive integer \( n \). If we endow the latter set with the topology of pointwise convergence (that is, the product topology), then there is no difference between \( \mathbb{R}^n \) and \( \mathbb{R}^{\{1,\ldots,n\}} \) from the viewpoint of topology either. Similarly, \( \mathbb{R}^\infty \) and \( \mathbb{R}^\mathbb{N} \) are the same topological spaces.

In the remainder of this text, when we view \( Y^X \) as a “function space,” we will refer to the product topology on \( Y^X \) as the **topology of pointwise convergence** (where \( X \) is any nonempty set and \( Y \) a topological space).
Exercises

1.1. Let $\mathcal{F}$ stand for the set of all $\{0,1\}$-valued functions $f$ on $\mathbb{R}$ such that $f^{-1}(0)$ is a finite set. Let $\mathbf{0}$ denote the self-map on $\mathbb{N}$ that equals to zero everywhere.

a. Show that $\mathbf{0}$ is not in $\text{cl}(\mathcal{F})$, if we view $\mathcal{F}$ as residing in $\mathcal{B}(\mathbb{N})$.

b. Show that $\mathbf{0} \in \text{cl}(\mathcal{F})$, if we view $\mathcal{F}$ as residing in $\mathbb{R}^\mathbb{N}$ with the topology on $\mathbb{R}^\mathbb{N}$ being the topology of pointwise convergence.

1.2. (**The Box Topology**) Let $I$ be a nonempty set, $X_i$ a topological space for each $i \in I$, and $X$ the Cartesian product of $\{X_i : i \in I\}$. We have seen above that the box topology is finer than the product topology on $X$, while the two topologies are the same when $I$ is finite. Despite its intuitively appealing definition, however, the box topology is “too fine” to be useful when it comes to matters of convergence. We give two illustrations of this fact in this exercise.

a. Show that the map $f : \mathbb{R} \to \mathbb{R}^\infty$, defined by $f(x) := (x,x,\ldots)$, is continuous when we view $\mathbb{R}^\infty$ as a topological space relative to the product topology, but not relative to the box topology. Thus, the “if” part of the observation we have noted in Example 1.12 is false for the box topology. (Hint: Is the inverse image of $(-1,1) \times (-\frac{1}{2},\frac{1}{2}) \times \ldots$ open under $f$?)

b. Show that the sequence $(x_m)$ in $\mathbb{R}^\infty$, where $x_m := (0,\ldots,0,m,m,\ldots)$ with the first nonzero term being the $m$th one, converges relative to the product topology, but not relative to the box topology. Conclude that the “if” part of Proposition 1.5 fails for the box topology.

1.3. Let $X_1,\ldots,X_n$ be finitely many homeomorphic topological spaces, and let $X$ be defined as in Exercise 1.6 of Chapter 4. Consider $X$ as a topological space relative to the disjoint union topology, and show that $X \cong X_1 \times \{1,\ldots,n\}$ where $\{1,\ldots,n\}$ is given the discrete topology.

1.4. Prove: A topological space $X$ is discrete iff $\Delta_X$ is open in $X \times X$.

1.5. Prove: $\mathbb{D}^n \times \mathbb{D}^m \cong \mathbb{D}^{n+m}$ for any $m,n \in \mathbb{N}$.

1.6. Describe, geometrically, how you would embed $S^1 \times S^2$ and $S^1 \times S^1 \times [0,1]$ into $\mathbb{R}^3$.

1.7. For any positive integer $n$, prove that $O(n) \cong SO(n) \times \{-1,1\}$. (Hint: What happens to the determinant of a matrix if you multiply one of its rows by a constant?)

1.8. Let $n$ be a positive integer with $n \geq 2$. The $n$-fold product $S^1 \times \cdots \times S^1$ is called the $n$-dimensional torus. Prove that the $n$-dimensional torus can be embedded in $\mathbb{R}^{n+1}$.

1.9. Take finitely many positive integers $n_1,\ldots,n_k$, and let $X_i$ be an $n_i$-manifold for each $i = 1,\ldots,k$. (Recall Exercise 4.15 of Chapter 4.) Prove that $X_1 \times \cdots \times X_k$ is an $(n_1 + \cdots + n_k)$-manifold.
1.10. Let $X$ and $Y$ be topological spaces, and $f \in C(X,Y)$. Prove that $Y$ is Hausdorff iff \{(x_1, x_2) : f(x_1) = f(x_2)\} is closed in $X \times X$.

1.11. Let $X$ and $Y$ be topological spaces, and take any $f : X \to Y$. Consider the map $F : X \to \text{graph}(f)$, defined by $F(x) := (x, f(x))$, and show that $F$ is an homeomorphism iff $f \in C(X,Y)$.

1.12. Let $I$ be a nonempty set, $X_i$ a topological space for each $i \in I$, and $X$ the topological product of $\{X_i : i \in I\}$.

   a. For each $i \in I$, let $S_i$ be a subset of $X_i$, and show that the closure of $\prod_{i \in I} S_i$ in $X$ equals $\prod_{i \in I} \text{cl}(S_i)$.

   b. Show that the interior of $\prod_{i \in I} S_i$ in $X$ is nonempty only if $S_i = X_i$ for all but finitely many $i \in I$.

   c. Give an example to show that the interior of $\prod_{i \in I} S_i$ in $X$ need not equal $\prod_{i \in I} \text{int}(S_i)$. (Hint: $I$ must be infinite in this example.)

1.13. Let $I$ be a nonempty set, $X_i$ and $Y_i$ topological spaces for each $i \in I$, and $X$ and $Y$ the topological products of $\{X_i : i \in I\}$ and $\{Y_i : i \in I\}$, respectively. For each $i \in I$, let $\varphi_i : X_i \to Y_i$ be an open map, and define $\varphi : X \to Y$ by $\varphi(x)(i) := \varphi_i(x(i))$ for each $i \in I$. Show that if $\varphi_i$ is a surjection for all but finitely many $i \in I$, then $\varphi$ is an open map.

1.14. Let $I$ be a nonempty set, $X_i$ a topological space for each $i \in I$, and $X$ the topological product of $\{X_i : i \in I\}$.

   a. Take any subset $S_i$ of $X_i$ for each $i \in I$, and prove that $\prod_{i \in I} S_i$ is dense in $X$ iff $S_i$ is dense in $X_i$ for each $i \in I$.

   b. Take any $x^*_i \in X_i$ for each $i \in I$, and prove that $\{x \in X : x(i) = x^*_i \text{ for all but finitely many } i \in I\}$ is dense in $X$.

1.15. Let $I$ and $X$ be nonempty sets. For each $i \in I$, let $Y_i$ be a topological space and $f_i : X \to Y_i$ a function. Let $Y$ stand for the topological product of $\{Y_i : i \in I\}$, and define the map $f : X \to Y$ by $f(x)(i) := f_i(x)$.

   a. Prove that the weak topologies induced by $\{f_i : i \in I\}$ and $\{f\}$ on $X$ are the same.

   b. We say that $\{f_i : i \in I\}$ is a separating class for $X$ if for every distinct $x_1$ and $x_2$ in $X$, there is an $i \in I$ with $f_i(x_1) \neq f_i(x_2)$. Show that $f$ is an embedding, provided that $\{f_i : i \in I\}$ is a separating class for $X$.

2 Properties of the Product Topology

2.1 First/Second Countability of Product Spaces

There is no reason for the product of uncountably many topological spaces to be first-countable, even when the coordinate spaces are particularly well-behaved (being, say, metrizable). This is only to be expected. The good
news, which you might have already conjectured, is that all goes well on this account with countable products. That is, taking the topological product of countably many topological spaces preserves the first- and second-countability properties of the coordinate spaces. More generally, we have:

**Theorem 2.1.** Let \( I \) be a nonempty set, \( X_i \) a topological space for each \( i \in I \), and \( X \) the topological product of \( \{X_i : i \in I\} \). Then, \( X \) is first-countable (second-countable) if, and only if, (i) \( X_i \) is first-countable (second-countable) for each \( i \in I \), and (ii) \( X_i \) is an indiscrete space for all but countably many \( i \in I \).

An immediate consequence of this result is the “good news” we mentioned above:

**Corollary 2.2.** The product of countably many topological spaces is first-countable (second-countable) if, and only if, each coordinate space of the product is first-countable (second-countable).

Theorem 2.1 also tells us that the product of uncountably many first-countable topological spaces may fail to be first countable, no matter how well-behaved these spaces are. For instance, apparently, \( \mathbb{R}^{[0,1]} \), the set of all real maps on \( [0,1] \), is not first-countable relative to the topology of pointwise convergence. If only to make things a bit more concrete, we provide a direct confirmation of this fact here, before we prove Theorem 2.1.

**Example 2.1.** We wish to show that the product topology on \( \mathbb{R}^{[0,1]} \) is not first-countable. We do this in two steps.

[Step 1] Define

\[
S := \{x \in \mathbb{R}^{[0,1]} : x(i) = 1 \text{ for all but finitely many } i \in [0,1]\},
\]

and consider the map \( x_0 \in \mathbb{R}^{[0,1]} \) with \( x_0(i) = 0 \) for each \( i \in [0,1] \). Let us first verify that \( x_0 \) belongs to the closure of \( S \). Take any basis element \( O \) for the product topology on \( \mathbb{R}^{[0,1]} \) such that \( x_0 \in O \). By definition, there is a finite subset \( J \) of \( [0,1] \) such that \( O \) is the Cartesian product of a collection like \( \{O_i : i \in [0,1]\} \) where \( 0 \in O_j \in \mathcal{O}_\mathbb{R} \) for all \( j \in J \), and \( O_i = \mathbb{R} \) for all \( i \in [0,1] \setminus J \). Thus, the map \( x \in \mathbb{R}^{[0,1]} \) that equals 0 on \( J \) and 1 on \( [0,1] \setminus J \), belongs to \( S \cap O \). We conclude that every open neighborhood of \( x_0 \) intersects \( S \), which means \( x_0 \in \text{cl}(S) \).

[Step 2] We now go for the kill. To derive a contradiction, suppose \( \mathbb{R}^{[0,1]} \) is first-countable. Then, by Proposition 2.2 of Chapter 4, there is a
sequence \((x_m)\) in \(S\) that converges to \(x_0\). As each \(x_m\) belongs to \(S\), for each positive integer \(m\), we may have \(x_m(i) \neq 1\) for at most finitely many \(i\). Thus, \(\{i \in [0,1] : x_m(i) \neq 1\}\) is a countable subset of \([0,1]\). As \([0,1]\) is uncountable, then, there is a \(j\) in \([0,1]\) such that \(x_m(j) = 1\) for every positive integer \(m\). Consequently, \(U := \{x \in \mathbb{R}^{[0,1]} : x(j) < 1\}\) is an open neighborhood of \(x_0\) such that \(x_m\) does not belong to \(U\) for any \(m\), contradicting \(x_m \to x\).

The rest of this subsection is devoted to the proof of Theorem 2.1. (This proof is not really difficult, but reads a bit tedious due to its heavy notation.) Let us first put the following observation on record; this will help us prove the “only if” part of Theorem 2.1.

**Lemma 2.3.** Let \(X\) and \(Y\) be two topological spaces, and \(f : X \to Y\) a continuous and open map. If \(X\) is first-countable (second-countable), then so is \(f(X)\).

**Proof.** Suppose \(X\) is first-countable, and take any \(y \in f(X)\). Pick any \(x\) in \(X\) with \(y = f(x)\), and let \(B(x)\) be a countable (local) basis at \(x\), and define \(B'(y) := \{f(B) : B \in B(x)\}\). We leave as an exercise to check that \(B'(y)\) is a (local) basis at \(y\) (where the ambient space is the subspace \(f(X)\) of \(Y\)). As \(B'(y)\) is obviously countable and \(y\) was arbitrarily chosen in \(f(X)\), this proves that \(f(X)\) is first-countable. (The same argument also establishes the second-countability part of the lemma.)

Thus, a continuous and open image of a first-countable space is itself first-countable. In particular, as the projection maps are continuous and open surjections, this result implies that the coordinate spaces of any first-countable product space are themselves first-countable. (The same goes for second-countable spaces as well.)

**Proof of Theorem 2.1.** We only prove the part of the proposition about first-countability; the part about second-countability is proved analogously. Let us first assume that \(X\) is first-countable. As we have noted above, Lemma 2.3 entails that \(X_i\) is first-countable for each \(i \in I\). To complete our proof, let \(J\) stand for all indices \(j\) in \(I\) such that \(X_j\) is not indiscrete. Our goal is to show that \(J\) is countable. To this end, for each \(j \in J\), let us choose a point \(x_j^* \in X_j\) which has an open neighborhood in \(X_j\) other than \(X_j\). Let \(x\) be a point in \(X\) such that \(x(j) = x_j^*\) for each \(j \in J\). Since \(X\) is first-countable, there is a countable (local) basis \(B(x)\) at \(x\). By definition of the product topology, for any open neighborhood \(O\) of \(x\) in \(X\), we have
proj_i(O) = X_i for all but finitely many i \in I. Thus, for every B \in \mathcal{B}(x), there is a finite subset I(B) of I such that

\[
\text{proj}_i(B) \begin{cases} 
  X_i, & \text{if } i \in I \setminus I(B) \\
  X_i & \text{if } i \in I(B).
\end{cases}
\]

Put J' := \bigcup \{I(B) : B \in \mathcal{B}(x)\}. Being the union of countably many finite sets, J' is a countable set. We claim that J \subseteq J'. To see this, take any j \in J. By the choice of x, there is an open neighborhood O_j of x(j), distinct from X_j. As proj_j^{-1}(O_j) is an open neighborhood of x, then, there is a B \in \mathcal{B}(x) such that B \subseteq proj_j^{-1}(O_j), but this means that proj_j(B) \subseteq X_j, that is, j \in I(B) \subseteq J'. Thus: J \subseteq J'. As J' is countable, therefore, so is J, as we sought.

Conversely, let us now assume that each X_i is first-countable, and that there is a countable subset J of I such that X_i is an indiscrete space for each i \in J. Take any point x in X, and for each i \in I, let \mathcal{B}(x(i)) be a countable (local) basis at x(i) (in the space X_i). Obviously, \mathcal{B}(x(i)) = \{X_i\} for each i \in I \setminus J. We put

\[
V := \bigcup_{j \in J} \{\text{proj}_j^{-1}(B) : B \in \mathcal{B}(x(j))\}.
\]

As the union of countably many countable sets is countable, V is a countable collection of open neighborhoods of x in X. Let \mathcal{B} stand for the collection of all intersections of finitely many elements of V. As the set of all finite subsets of a countable set is countable, \mathcal{B} is a countable subset of \mathcal{O}_X(x) as well. We now claim that \mathcal{B} is a (local) basis at x in X. To see this, take any O \in \mathcal{O}_X(x). If O = X all is trivial, so we may assume that O \subset X. Furthermore, we may assume that O is an element of the basis for the product topology on X. Then, proj_i(O) = X_i for all but finitely many i \in I, say, i_1, ..., i_k. (Since O is not X, there is at least one such i_s.) Obviously, each of these indices belong to J. Besides, for each s = 1, ..., k, there is an open subset O_{i_s} of X_{i_s} such that

\[
x \in O = \bigcap_{s=1}^{k} \text{proj}^{-1}_{i_s}(O_{i_s}).
\]

For each s = 1, ..., k, we pick any B_{i_s} from \mathcal{B}(x(i_s)) such that x(i_s) \in B_{i_s} \subseteq O_{i_s}. Then, B := \bigcap_{s=1}^{k} \text{proj}_{i_s}^{-1}(B_{i_s}) belongs to \mathcal{B}, while x \in B \subseteq O. Given the arbitrary choice of O, we conclude that \mathcal{B} is a (local) basis at x. Our proof is now complete.
2.2 Metrizability of Product Spaces

We now look at the issue of metrizability of the product of a collection of metrizable topological spaces. Again, all goes well in the case of countable products.

**Proposition 2.4.** The topological product of countably many metrizable spaces is metrizable.

**Proof.** Take any nonempty countable set \( I \), and let \( X_i \) be a metrizable space for each \( i \in I \). Let \( X \) be the topological product of \( \{ X_i : i \in I \} \). For each \( i \in I \), let \( d_i \) be a metric on \( X_i \) that metrizes the topology of \( X_i \). We set \( I^* \) as \( \{1, \ldots, |I|\} \) if \( I \) is finite, and as \( \mathbb{N} \) if \( I \) is infinite. We may then enumerate \( I \) as \( \{i_k : k \in I^*\} \), and define the real map \( d_X \) on \( X \times X \) by

\[
    d_X(x, y) := \sum_{k \in I^*} 2^{-k} \min\{1, d_{i_k}(x(i_k), y(i_k))\},
\]

which is easily checked to be a metric on \( X \). We wish to show that \( d_X \) metrizes the product topology on \( X \).

Let \( O \) be an element of the basis for the product topology on \( X \). Then, there is a finite subset \( K \) of \( I^* \) such that \( O \) is the Cartesian product of a collection \( \{O_i : i \in K\} \) where \( O_{i_k} \subseteq O_{X_k} \) for each \( k \in K \), and \( O_{i_k} = X_k \) for all \( k \in I^* \setminus K \). Now take any \( x \) in \( O \), and note that if we choose \( \varepsilon \in (0, 1) \) small enough that \( B_{X_{i_k}}(x(i_k), 2^k \varepsilon) \subseteq O_{i_k} \) and \( \varepsilon < 2^{-k} \) for each \( k \in K \), we have \( B_X(x, \varepsilon) \subseteq O \). (Indeed, for any \( y \in X \) and \( k \in K \), if \( d_X(x, y) < \varepsilon \) holds, then we have \( 2^{-k} \min\{1, d_{i_k}(x(i_k), y(i_k))\} < \varepsilon \), and hence, \( d_{i_k}(x(i_k), y(i_k)) < 2^k \varepsilon \), that is, \( y \in O \).) It follows that \( O \) is open in \( (X, d) \). It follows that \( O \) is open in the metric space \( (X, d) \). In view of the arbitrariness of the choice of \( O \), therefore, we conclude: \( O_X \subseteq O_{(X, d)} \).

 Conversely, take any \( O \in O_{(X, d)} \). Fix an \( x \) in \( O \) and pick any \( \varepsilon > 0 \) with \( B_X(x, \varepsilon) \subseteq O \). As \( \sum_{i=1}^{\infty} 2^{-i} \) converges, there is a positive integer \( m \) such that \( \sum_{i=m+1}^{\infty} 2^{-i} < \varepsilon/2 \). We set \( I^* := \{k \in I^* : k \leq m\} \), and

\[
    \varepsilon_k := \frac{\varepsilon}{2(2^{-1} + \ldots + 2^{-k})} \quad \text{for each } k = 1, \ldots, m.
\]

Now put

\[
    O_{i_k} := \begin{cases} 
        B_{X_i}(x(i_k), \varepsilon_k), & \text{if } k \leq m \\
        X_{i_k}, & \text{otherwise},
    \end{cases}
\]

and consider the product \( U \) of \( \{O_{i_k} : k \in I^*\} \). Clearly, \( U \) is an element of the basis for the product topology on \( X \). Furthermore, \( x \in U \subseteq B_X(x, \varepsilon) \)
because \( y \in U \) implies

\[
d_X(x, y) \leq \sum_{k \in I^{**}} 2^{-k} d_{i_k}(x(i_k), y(i_k)) + \frac{\varepsilon}{2} < \sum_{k \in I^{**}} 2^{-k} \varepsilon_k + \frac{\varepsilon}{2} < \varepsilon.
\]

We may thus conclude that \( O \) is an open subset of \( X \). Conclusion: \( O_X \supseteq O_{(X, d)} \).

**Remark.** The metric we chose to metrize \( X \) above is clearly motivated by how we have metrized the product of countably many metric spaces earlier in Section 3.2 of Chapter 1. Just as it was the case there, we have bounded the metrics \( d_i \) above by taking their composition with the map \( t \mapsto \min\{1, t\} \), and used a weighting scheme that fast converges to 0, not to run into summability problems. In the case of the products of finitely many spaces such problems do not arise, so we can use simpler metrizations. Indeed, were \( I \) known to be finite, we could simply define \( d_X(x, y) := \sum_{i \in I} d_i(x(i), y(i)) \) in the proof above.

A natural question at this point is if we can also metrize the product of uncountably many metrizable spaces. Unfortunately, in general, this is not possible to do. Indeed, since metrizability implies first-countability (Proposition 2.3 of Chapter 4), and an indiscrete space is metrizable only if it contains a single point, the following is an immediate consequence of Theorem 2.1.

**Corollary 2.5.** Let \( I \) be a nonempty set, \( X_i \) a topological space for each \( i \in I \), and \( X \) the topological product of \( \{X_i : i \in I\} \). If \( X \) is metrizable, then \( |X_i| = 1 \) for all but countably many \( i \in I \).

So, for instance, the product topology on \( \mathbb{R}^{[0,1]} \) is not metrizable, something we already knew from Example 2.1.

### 2.3 Separability of Product Spaces

We now turn to the matter of separability of the topological product of a given collection of separable topological spaces. In this case, the results are unexpectedly pleasant. For one thing, we certainly have a positive result if that collection is countable. This is fairly expected. Indeed, Proposition 7.3 of Chapter 1 states that the product of countably many separable metric spaces is separable. The following result provides a generalization:
Proposition 2.6. The topological product of countably many separable topological spaces is separable.

Proof. Take any nonempty countable set \( I \), and let \( X_i \) be a separable topological space for each \( i \in I \). Let \( X \) be the topological product of \( \{ X_i : i \in I \} \). For each \( i \), let \( Z_i \) be a countable dense subset of \( X_i \), and pick an element \( z_i \) from \( Z_i \). We define

\[
Z := \{ x \in X : x(i) = z_i \text{ for all but finitely many } i \in I \}.
\]

Then, \( Z \) is a countable set (because the set of all finite subsets of a countable set is countable). We claim that \( Z \) is dense in \( X \). Thus, to complete our proof, it is enough to show that \( Z \) intersects any basis element of the product topology on \( X \). To this end, let \( O \) be such a basis element, that is, let \( O = \bigcap \{ \text{proj}_j^{-1}(O_j) : j \in J \} \), where \( J \) is a nonempty finite subset of \( I \), and \( O_j \) is an open subset of \( X_j \), distinct from \( X_j \), for each \( j \in J \). Being dense in \( X_j \), the set \( Z_j \) intersects \( O_j \), that is, there is a point \( y(j) \) in \( O_j \cap Z_j \), for each \( j \in J \). Now define \( x \in X \) as

\[
x(i) := \begin{cases} 
y(j), & \text{if } j \in J, 
z_i, & \text{otherwise.}
\end{cases}
\]

Clearly, \( x \in O \cap Z \). In view of the arbitrary choice of \( O \), therefore, we may conclude that \( Z \) is dense in \( X \).

Unlike the case with, say, first-countability, it turns out that we can actually do quite a bit better than this. Indeed, we will shortly prove that even the product of uncountably many separable spaces (which are not indiscrete) may turn out to be separable. Let us first agree on some terminology. In what follows, we say that a set \( S \) has at most the cardinality of the continuum, if there is an injection from \( S \) into \( \mathbb{R} \). (This is often denoted by writing \( |S| \leq c \).) What we intend to prove here is this:

Theorem 2.7. Let \( I \) be a nonempty set, \( X_i \) a separable topological space for each \( i \in I \), and \( X \) the topological product of \( \{ X_i : i \in I \} \). If \( I \) has at most the cardinality of the continuum, then \( X \) is separable.

For instance, surprisingly, it follows that \( \mathbb{R}^{[0,1]} \) is a separable topological space relative to the product topology while it is certainly not first-countable (Theorem 2.1). Theorem 2.7 is known in the folklore as the Hewitt-Marczewski-Pondiczery Theorem (but note that Pondiczery is only a pseudonym for Ralph P. Boas Jr. who wrote many mathematical
articles under various pseudonyms). Our proof is based on the following purely set-theoretic observation.

Lemma 2.8. Let $I$ be a nonempty set that has at most the cardinality of the continuum. Then, there exists a countable collection $\mathcal{I}$ of subsets of $I$ with the following property: For every positive integer $k$, and every $i_1, \ldots, i_k \in I$, there exist pairwise disjoint sets $I_1, \ldots, I_k \in \mathcal{I}$ such that $i_1 \in I_1, \ldots, i_k \in I_k$.

Proof. By hypothesis, there is an injection $f$ from $S$ into $\mathbb{R}$. We define

$$\mathcal{I} := \{f^{-1}((a, b)) : a, b \in \mathbb{Q}\}.$$ 

Clearly, $\mathcal{I}$ is countable. To see that $\mathcal{I}$ satisfies the required property, take any $k \in \mathbb{N}$ and $i_1, \ldots, i_k \in I$. Then, $f(i_1), \ldots, f(i_k)$ are distinct real numbers, so we can find rational numbers $a_1, b_1, \ldots, a_k, b_k$ such that $(a_1, b_1), \ldots, (a_k, b_k)$ are pairwise disjoint intervals such that $f(i_1) \in (a_1, b_1), \ldots, f(i_k) \in (a_k, b_k)$. Then, where $I_i := f^{-1}((a_i, b_i))$ for each $i$, the sets $I_1, \ldots, I_k$ are pairwise disjoint subsets of $\mathcal{I}$ with $i_1 \in I_1, \ldots, i_k \in I_k$.

Proof of Theorem 2.7. Assume that $I$ has at most the cardinality of the continuum, and let $\mathcal{I}$ be as found in Lemma 2.8. For each $i \in I$, let $Z_i$ be a countable dense subset of $X_i$, which we enumerate as $\{z_{i,1}, z_{i,2}, \ldots\}$. For any positive integer $n$, let $S_n$ be the set of all $2n$-tuples $(I_1, \ldots, I_n, l_1, \ldots, l_n)$ where $I_1, \ldots, I_n$ are pairwise disjoint elements of $\mathcal{I}$ and $l_1, \ldots, l_n$ are positive integers. Finally, put $\mathcal{S} := S_1 \cup S_2 \cup \cdots$, which is clearly a countable set.

Let us now pick any point $x_i^*$ in $X_i$ for each $i \in I$. Next, for every positive integer $k$ and $2k$-tuple $s := (I_1, \ldots, I_k, l_1, \ldots, l_k)$ in $S_k$, we define $x_s \in X$ as

$$x_s(i) := \begin{cases} z_{i,l_j}, & \text{if } i \in I_k \text{ for some } j = 1, \ldots, k, \\ x_i^*, & \text{otherwise.} \end{cases}$$

Finally, we put $Z := \{x_s : s \in \mathcal{S}\}$. Clearly, $Z$ is a countable subset of $X$. We will complete our proof by showing that this set is dense in $X$. To do this, it is enough to show that $Z$ intersects every element of the basis for the product topology on $X$. Take, then, any open subset $O$ of $X$ of the form

$$O := \bigcap_{j=1}^k \text{proj}^{-1}_i(O_j)$$

where $k \in \mathbb{N}$ and $O_j$ is any nonempty open subset of $X_{i_j}$, $j = 1, \ldots, k$. Choose any pairwise disjoint $I_1, \ldots, I_k$ in $\mathcal{I}$ so that $i_1 \in I_1, \ldots, i_k \in I_k$. Owing to its denseness, $Z_{ij}$ intersects $O_{i_j}$ for each $j = 1, \ldots, k$. For any $j$, then,
there is a point $z_{i,j}$ in $O_{i,j} \cap Z_{i,j}$. Then, for $s = (I_1, \ldots, I_k, l_1, \ldots, l_k)$, we have $x_s(i) = z_{i,j} \in O_j$ for each $j = 1, \ldots, k$, so $x_s \in O$, that is, $Z$ intersects $O$. Our proof is now complete.

A natural question now is if Theorem 2.7 is a first-best result. That is: Is the product of a collection of separable topological spaces separable, even when the cardinality of that collection exceeds that of the continuum? We next show that the answer is no, at least for Hausdorff spaces.

**Proposition 2.9.** Let $I$ be a nonempty set, $X_i$ a nonsingleton separable Hausdorff space for each $i \in I$, and $X$ the topological product of $\{X_i : i \in I\}$. If $X$ is separable, then $I$ has at most the cardinality of the continuum.

**Proof.** For each $i \in I$, let us pick a nonempty open proper subset of $X_i$, say, $O_i$. (There is such a set, because $X_i$ is a Hausdorff space that contains at least two points.) Suppose that $X$ is separable, and take any countable dense subset $Z$ of $X$. For each $i \in I$, we define

$$Z_i := \{x \in X : x(i) \in O_i\}.$$  

Clearly, $Z_i$ is open in $X$, so it must intersect $Z$; let us denote this nonempty intersection by $Y_i$, for each $i \in I$. Notice that $Y_i \neq Y_j$ for any distinct $i$ and $j$ in $I$. (Why?) It follows that $i \mapsto Y_i$ is an injection from $I$ into $2^Z$. But, as $Z$ is countable, we have $\mathbb{R} \cong \text{card}2^Z$, so we conclude that $I$ has at most the cardinality of the continuum.

3 The Quotient Topology

3.1 First Impressions

Our task in this section is to have a look at a method of topologizing a given partition of a topological space. This issue arises in a variety of contexts, and the method we will outline is widely used in algebraic topology. Even though we will not get into algebraic topology in any significant way in this text, we too will utilize this method on separate occasions.

Let $X$ be a topological space, and $\sim$ an equivalence relation on $X$. The **quotient map** on $X$ induced by $\sim$ is the surjection $x \mapsto [x]_\sim$. We will denote this function here by $\pi_\sim$, that is, $\pi_\sim : X \to X/\sim$ is the map defined by

$$\pi_\sim(x) := [x]_\sim.$$
The idea here is to think of all elements of the set \([x]_\sim\) as “one.” In other words, we would like to “identify” all elements that are declared equivalent by \(\sim\). Speaking set-theoretically, this leads to the partition \([\{[x]_\sim : x \in X\}\)] of \(X\), which we denote by \(X/\sim\). In addition, we would like this collection to have a topology that is somehow loyal to the original topology of \(X\). At the very least, we should topologize \(X/\sim\) in such a way that our process of identification, that is, the quotient map \(x \mapsto [x]_\sim\), is continuous. (If, for instance, the topology of \(X\) is metric, we would like nearby points to belong to nearby cells in the partition \(X/\sim\).) Clearly, the more open sets this topology has, the harder it becomes for \(\pi_\sim\) to be continuous, and the more stringent this requirement becomes. Fortunately, there is a largest (finest) topology on \(X/\sim\) that meets this condition.

**Definition.** Let \(X\) be a topological space, and \(\sim\) an equivalence relation on \(X\). The quotient topology on the partition \(X/\sim\) of \(X\) is the collection

\[ O_{X/\sim} := \{ O \in 2^{X/\sim} : \pi^{-1}_\sim(O) \in O_X \} \]

(with the understanding that \(\pi^{-1}_\sim(\emptyset) = \emptyset\)). That is, a subset \(O\) of \(X/\sim\) is open relative to the quotient topology iff \(\bigcup O\) is open in \(X\) (with the understanding that \(\bigcup \emptyset = \emptyset\)).

As inverse images preserve intersections and unions, \(O_{X/\sim}\) is indeed a topology on \(X/\sim\). Furthermore, it is plain from its definition that the quotient topology on \(X/\sim\) is the largest among all topologies on \(X/\sim\) that renders the quotient map \(\pi_\sim\) continuous. In other words,

\[ \pi^{-1}_\sim(O) \text{ is open in } X \iff O \text{ is open in } X/\sim, \]

that is, a set \(O\) in \(X/\sim\) is open relative to the quotient topology iff its inverse image under the quotient map on \(X\) is open in \(X\). (For, \(\pi^{-1}_\sim(O) = \bigcup O\) for any subset of \(X/\sim\).) Not only is it fairly natural, we will see shortly that this topology is abundantly useful.

**Remark.** The characterization of the quotient topology in terms of closed sets mirrors that we have given above: A subset \(C\) of \(X/\sim\) is closed iff \(\bigcup C\) is closed in \(X\). To see this, put \(O := (X/\sim) \setminus C\), and observe that \(O\) is open in \(X/\sim\) iff \(\bigcup O\) is open in \(X\), that is, \(X \setminus \bigcup O\) is closed in \(X\). As \(X \setminus \bigcup O = \bigcup C\), our assertion follows.

**A First Set of Examples**
It usually takes a bit of an effort to get comfortable with the quotient topology. We thus begin our work here by looking at some simple examples.

**Example 3.1.** Let $X$ be a topological space, and let $\sim$ be the equivalence relation on $X$ defined as $x \sim y$ iff $x = y$. (That is, $\sim$ equals $\Delta_X$.) Then, $\pi_\sim$ is nothing but the map $x \mapsto \{x\}$, so, in this (very special) case $\pi_\sim$ is a homeomorphism between $X$ and $X/\sim$. Thus, trivially, identifying every element of $X$ with itself does not at all alter the topological structure of $X$.

Let us now look at the other extreme. Suppose $\pi_\sim$ is the equivalence relation on $X$ defined as $x \sim y$ iff $x, y \in X$. (That is, $\sim$ equals $X \times X$.) Then, $\pi_\sim$ is the constant map $x \mapsto X$, which “identifies” all members of $X$ with each other. Naturally, this leads to a singleton topological space. That is, $X/\sim = \{X\}$, and the quotient topology on $X/\sim$ is, trivially, $\{\emptyset, \{X\}\}$.

**Example 3.2.** Let $X$ be a topological space such that there is a nonempty proper subset $O$ of $X$ which is open, but not closed, in $X$. Let $\pi_\sim$ be the equivalence relation on $X$ defined as $x \sim y$ iff either $\{x, y\} \subseteq O$ or $\{x, y\} \subseteq X \setminus O$. Then, $\pi_\sim$ identifies everything in $O$ with each other, and everything outside of $O$ with each other. That is,

$$
\pi_\sim(x) = \begin{cases} 
O & \text{if } x \in O \\
X \setminus O & \text{otherwise},
\end{cases}
$$

and hence $X/\sim = \{O, X \setminus O\}$. Furthermore, by the choice of $O$, we have $O_{X/\sim} = \{\emptyset, \{O\}, X/\sim\}$, that is, $X/\sim$ is homeomorphic to the Sierpiński space.

**Example 3.3.** Consider the equivalence relation $\sim$ on $\mathbb{R}$ defined by $x \sim y$ iff $x - y \in \mathbb{Z}$. Then, the quotient space $\mathbb{R}/\sim$ is obtained from $\mathbb{R}$ by identifying any given real number with all of its multiples. (Indeed, we have $\pi_\sim(x) = \{kx : k \in \mathbb{Z}\setminus\{0\}\}$ in this example.) We will see below that this quotient space is homeomorphic to $S^1$.

**Example 3.4.** Let $n$ be any positive integer, and consider the equivalence relation $\sim$ on $X := \mathbb{R}^n \setminus \{0\}$ defined by $x \sim y$ iff $\alpha x = y$ for some nonzero real number $\alpha$. (Here, as usual, $0$ stands for the $n$-vector of $0$s.) Then, $\pi_\sim(x)$ equals the line that goes through $0$ and $x$, but with $0$ deleted. So, as a set, $X/\sim$ can be identified with the real projective space $\mathbb{R}P^{n-1}$. In fact, if you recall how we defined the topology of $\mathbb{R}P^{n-1}$ in Example 1.5 of Chapter 4, you will see immediately that $\mathbb{R}P^{n-1}$ and $X/\sim$ are the same topological spaces. (Of course, by “the same” here, we mean “homeomorphic,” but in
this case the homeomorphism between these spaces is the trivial one that maps any line $l$ in $\mathbb{R}P^{n-1}$ to $l\backslash \{0\}$."

This is not the only way in which we can think of $\mathbb{R}P^{n-1}$ as a quotient space. Another way of doing this is to use the $(n-1)$-sphere. Indeed, we have

$$\mathbb{R}P^{n-1} \cong S^{n-1}/\sim,$$

where $\sim$ is the equivalence relation on $S^{n-1}$ defined as $x \sim y$ if $x = \mp y$. (Here, $\pi_\sim$ identifies the antipodal points on the $(n-1)$-sphere, that is, $\pi_\sim(x) = \{-x, x\}$ for any $x \in S^{n-1}$.) The proof is straightforward.

**Quotient Spaces Determined by a Map**

Let $X$ be a topological space and $Y$ a nonempty set. There is a natural way of obtaining a quotient space from $X$ by using a given function $f : X \to Y$ in which we identify the points in $X$ whose images under $f$ are the same. More precisely, let us define the binary relation $\sim_f$ on $X$ by

$$x \sim_f y \iff f(x) = f(y),$$

which is clearly an equivalence relation. The quotient space $X/\sim_f$ is called the **quotient space determined by** $f$; we denote this topological space more succinctly as $X/f$.

**Remark.** This is actually a universal way of looking at quotient spaces. That is, every quotient space $X/\sim$ is, in fact, the quotient space determined by a map from $X$ into some nonempty set $Y$. Indeed, for any topological space $X$ and an equivalence relation $\sim$ on $X$, the spaces $X/\sim$ and $X/\pi_\sim$ are one and the same.

We can say more about the quotient space $X/f$ determined by $f : X \to Y$, if we know more about $f$. In particular, if $Y$ is a topological space and $f$ is continuous, then the map $F : X/f \to Y$, defined by

$$F([x]_\sim) := f(x),$$

is a (well-defined) continuous injection. *(Proof. That $F$ is well-defined and injective follows from the fact that $X/f$ is a partition of $X$. On the other hand, $f = F \circ \pi_\sim$, so given that $f$ is continuous, $\pi_\sim^{-1}(F^{-1}(O))$ is open in $X$, and hence by $(2)$, $F^{-1}(O)$ is open in $X/f$, for any $O \in O_Y$.) This map is actually a homeomorphism when $f$ is an open (or closed) map.*
Proposition 3.1. Let $X$ and $Y$ be topological spaces, and $f \in \mathcal{C}(X, Y)$. If $f$ is either open or closed, then $F : X/f \to f(X)$, defined by (3), is a homeomorphism.

Proof. (Throughout this proof, we write $[x]$ for $[x]_f$ to simplify our notation.) Suppose first that $f$ is open, and take any open set $O$ in $X/f$. By definition of the quotient topology, $O := \bigcup O$ is open in $X$. Moreover, by definition of $F$, we have $F(O) = \bigcup \{f(x) : [x] \in O\} = f(O)$. But, as $f$ is open, $f(O)$ is an open subset of $Y$. Therefore, $F(O)$ is open in $Y$ (and hence in $f(X)$). Conclusion: $F$ is a homeomorphism from $X/f$ onto $f(X)$.

Suppose next that $f$ is closed, and this time take any closed set $C$ in $X/f$. As we have remarked above, this means that $C := \bigcup C$ is closed in $X$. Therefore, repeating the argument we gave in the previous paragraph shows that $F(C)$ is a closed subset of $Y$ (and hence of $f(X)$). Thus, $F$ is a closed and continuous bijection, that is, a homeomorphism, from $X/f$ onto $f(X)$.

An immediate consequence of this proposition is that if $f$ is a continuous surjection from a topological space $X$ onto a topological space $Y$, which is either open or closed, then $Y$ can be thought of as obtained from $X$ (up to an homeomorphism) by identifying those points of $X$ that have the same images under the map. This observation is the basis of great many topological constructions. The following examples, and the ones we will consider in the next section, will provide several illustrations of this.

Example 3.5. Consider the equivalence relation $\sim$ on $\mathbb{R}$ defined by $x \sim y$ if $x = y$ or $\{x, y\} \subseteq [0, 1]$. Intuitively, $\mathbb{R}/\sim$ is the space obtained from $\mathbb{R}$ by shrinking the interval $[0, 1]$ to a point. We can think of this space as the quotient space determined by a variety of maps. In particular, $\mathbb{R}/f$ is precisely the same space as $\mathbb{R}/\sim$, where $f : \mathbb{R} \to \mathbb{R}$ is defined as

$$f(x) = \begin{cases} x, & \text{if } x < 0, \\ 0, & \text{if } 0 \leq x \leq 1 \\ 1 - x, & \text{if } 1 < x. \end{cases}$$

It is easily verified that $f$ is a continuous surjection which is closed. By Proposition 3.1, therefore, $\mathbb{R}/\sim \cong \mathbb{R}$.

Example 3.6. Consider the equivalence relation $\sim$ on $[0, 1]$ defined by $x \sim y$ if $x = y$ or $\{x, y\} \subseteq \{0, 1\}$. In words, $[0, 1]/\sim$ is obtained from $[0, 1]$ by identifying the endpoints of this interval. Geometrically speaking, we can think of $[0, 1]/\sim$ as obtained by bending the unit interval smoothly until we
make the endpoints meet, thereby making a loop. (In particular, the points
that are close to 0 and 1 are rendered close to each other in the resulting
space.) Of course, for this interpretation to hold water, we need to prove
something like $[0, 1]/\sim \cong S^1$. (Please note that this is not trivial. The space
on the left carries the quotient topology, while the one on the right is a
subspace of $\mathbb{R}^2$.)

We can again analyze the situation by viewing $[0, 1]/\sim$ as a quotient
space determined by a suitable map. To this end, we define $f : [0, 1] \to S^1$
by
$$f(x) := (\cos 2\pi x, \sin 2\pi x).$$
Then, $[0, 1]/\sim$ is the same space as $[0, 1]/f$. Moreover, it is readily checked
that $f$ is a continuous surjection which is open. By Proposition 3.1, therefore, $[0, 1]/\sim \cong S^1$.

**Remark.** The intuition behind the construction given in the previous ex-
ample extends to the entire real line. That is, we can think of the quotient
space $\mathbb{R}/\sim$, where $x \sim y$ iff $x - y \in \mathbb{Z}$, as obtained by wrapping the real
line around $S^1$ (countably) infinitely many times in a way that all integers
meet at the point $(0, 1)$ of $S^1$. Indeed, a minor modification of the argument
above shows that $\mathbb{R}/\sim \cong S^1$ in this case.

**Example 3.7.** Here is another modification of Example 3.6. Consider the
equivalence relation $\sim$ on $[0, 2]$ defined by $x \sim y$ iff $x = y$ or \{x, y\} $\subseteq$
$\{0, 1, 2\}$. Then, we can think of $[0, 2]/\sim$ as obtained from $[0, 2]$ by bending
this interval smoothly in a way that (only) the points 0, 1 and 2 meet,
thereby making a figure eight (in $\mathbb{R}^2$). In this case, then, our claim is that
$[0, 2]/\sim$ is homeomorphic to two circles with a single common point. This is
certainly true, but we leave the formal demonstration as an exercise.

**The Space $X/S$**

Examples 3.5, 3.6 and 3.7 are special cases of a general construction in which
we pick a certain nonempty subset $S$ of a given topological space $X$, and
identify all points in $S$ with each other, leaving all other points of $X$ intact.
We denote the resulting quotient space in what follows as $X/S$.

**Warning.** We use the notation $X/S$ instead of $X/S$, not to get into a conflict with
the standard way in which quotient groups are denoted in algebra. For instance,$\mathbb{R}/\mathbb{Z}$ is the quotient set induced by the equivalence relation $\sim$ on $\mathbb{R}$ defined by $x \sim y$
iff $x = y$ or $\{x, y\} \subseteq \mathbb{Z}$. By contrast, you will recall from group theory that $\mathbb{R}/\mathbb{Z}$
stands for the quotient set $\mathbb{R}/\sim$, where $x \sim y$ iff $x - y \in \mathbb{Z}$. (As topological spaces,
these are very different; the first one is countably infinitely many circles with a single common point, while the second one is the unit circle (up to homeomorphism.)

It is possible to examine the space $X/S$ from a general viewpoint as a disjoint union of $X\setminus S$ with a singleton space. To see this, pick any one point (in some space), say, $\emptyset$, that does not belong to $X$, and put $Y := (X\setminus S) \cup \{\emptyset\}$. Now define $f : X \to Y$ by $f(x) := x$ for $x \in X\setminus S$, and $f(x) := \emptyset$ for $x \in S$, and note that $X/S$ is none other than $X/f$. As we have seen in Examples 3.5 and 3.6, however, we can realize $X/S$ as a quotient space determined by a map by choosing a different function than this general one. Which map would be the best one to utilize in this sense depends on the context. The following example provides yet another illustration of this.

**Example 3.8.** $D^2/S^1$ is the quotient space obtained from the unit disc $D^2$ by identifying all the points in the boundary of $D^2$ with each other. Geometrically speaking, we can think of this space as obtained from $D^2$ by pulling a drawcord around the perimeter of $D^2$ tight enough so that the entire perimeter is bunched to a single point (in $\mathbb{R}^3$). And sure enough, this interpretation is formalized as: $D^2/S^1 \cong S^2$.

We may prove this as follows. First, note that $D^2/S^1 \cong \mathbb{R}^2$; indeed, $h : D^2/S^1 \to \mathbb{R}^2$, defined by $h(x) := \frac{x}{1-\|x\|^2}$, is a homeomorphism. Furthermore, where $z$ stands for the 3-vector $(0,0,1)$, we know that $\mathbb{R}^2$ and $S^2 \setminus \{z\}$ are homeomorphic. (See Example 3.13 of Chapter 2.) Let $g$ be a homeomorphism from $\mathbb{R}^2$ onto $S^2 \setminus \{z\}$. We now define $f : D^2 \to S^2$ by

$$f(x) = \begin{cases} g(h(x)), & \text{if } x \in D^2 \setminus S^1, \\ z, & \text{if } x \in S^1. \end{cases}$$

It is routine to check that $f$ is a continuous surjection which is closed. By Proposition 3.1, therefore, $D^2/S^1 \cong S^2$.

### 3.2 Gluing: Quotient Representation of Special Spaces

Quotient spaces determined by a function provides a method of obtaining a new topological space from a given space by identifying certain pairs of points in the space. From a geometric perspective, this process is best thought of as “gluing” certain points in the space to some other points, thereby bending the original space in a smooth manner to obtain a new space. We have in fact seen a simple example of this process already in Example 3.6 where we have “glued” the endpoints of the unit interval to
obtain the unit circle (up to homeomorphism). In this section, we look at some other examples of this sort.

**Example 3.9. (The Cylinder, Again)** We have defined the cylinder \( C \) in Example 1.8, and characterized it as \( S^1 \times [0, 1] \) (up to homeomorphism). Alternatively, we can obtain the cylinder as a certain quotient of the rectangle \([0, 1]^2\) by gluing the bottom of this rectangle to its top. More precisely, let \( \sim \) be the smallest equivalence relation on \([0, 1]^2\) such that \((s, 0) \sim (s, 1)\) for each \( s \in [0, 1] \). (That is, \( x \sim y \) iff \( x = y \), or \( x_1 = y_1 \) and \( x_2 = 0 = 1 - y_2 \).) Then, \([0, 1]^2/\sim \cong S^1 \times [0, 1] \). Indeed, \([0, 1]^2/\sim \) is none other than \([0, 1]^2/f\), where \( f : [0, 1]^2 \to S^1 \times [0, 1] \) is defined by

\[
f(s, t) := (\cos 2\pi t, \sin 2\pi t, s).
\]

It is readily checked that \( f \) is a continuous surjection which is open. By Proposition 3.1, therefore, \([0, 1]^2/\sim \cong S^1 \times [0, 1] \).

**Example 3.10. (The Torus, Again)** We have defined the 2-dimensional torus \( T^2 \) in Example 1.10, and characterized it as \( S^1 \times S^1 \) (up to homeomorphism). Alternatively, we can obtain \( T^2 \) as a certain quotient of the rectangle \([0, 1]^2\) by gluing the bottom of this rectangle to its top and its left side to its right side. More precisely, let \( \sim \) be the smallest equivalence relation on \([0, 1]^2\) such that \((s, 0) \sim (s, 1)\) and \((0, t) \sim (1, t)\) for each \( s, t \in [0, 1] \). Then, \([0, 1]^2/\sim \cong S^1 \times S^1 \). Indeed, \([0, 1]^2/\sim \) is none other than \([0, 1]^2/f\), where \( f : [0, 1]^2 \to S^1 \times S^1 \) is defined by

\[
f(s, t) := (\cos 2\pi s, \sin 2\pi s, \cos 2\pi t, \sin 2\pi t) .
\]

It is readily checked that \( f \) is a continuous surjection which is open. By Proposition 3.1, therefore, \([0, 1]^2/\sim \cong S^1 \times S^1 \).

**Remark.** We can think of the process of gluing outlined above for the torus iteratively. That is, we can first glue the bottom and top of the rectangle \([0, 1]^2\), thereby obtaining the cylinder as in Example 3.9. Then, we glue the left end of this cylinder (which is a circle) to its right end, thereby obtaining \( T^2 \). This way of doing things is sure to yield the same result with what we have done in Example 3.10, because the process of “taking quotients” is transitive. See Exercise 3.15 for the formalization of this point.

The previous two examples provide new characterizations of two familiar topological spaces by means of gluing. In the next two examples, we obtain new spaces by this method.
Example 3.11. (The Möbius Strip) Let $\sim$ be the smallest equivalence relation on $[0, 1]^2$ such that $(0, t) \sim (1, 1 - t)$ for each $t \in [0, 1]$. Then, the quotient space $[0, 1]^2/\sim$, which is called the **Möbius strip**, correspond to the surface in $\mathbb{R}^3$ which is constructed by gluing the left side off the rectangle $[0, 1]^2$ to its right side after giving a half twist.

Example 3.12. (The Klein Bottle) Let $\sim$ be the smallest equivalence relation on $[0, 1]^2$ such that $(s, 0) \sim (1 - s, 1)$ and $(0, t) \sim (1, t)$ for each $s, t \in [0, 1]$. Then, the quotient space $[0, 1]^2/\sim$, which is called the **Klein bottle**, corresponds to the space obtained from a cylinder by identifying the opposite ends with the orientation of the two circles reversed. The resulting space cannot be represented in $\mathbb{R}^3$ without self-intersection, but it can be embedded in $\mathbb{R}^4$. Proving the first of these claims require tools from algebraic topology that are beyond the scope of this text. Instead, we prove here the second claim.

Consider the following real maps on $[0, 1]^2$:

$$f_1(s, t) := (2 + \cos 2\pi s) \cos 2\pi t, \quad f_2(s, t) := (2 + \cos 2\pi s) \sin 2\pi t,$$

and

$$f_3(s, t) := (\sin 2\pi s) \cos \pi t, \quad f_4(s, t) := (\sin 2\pi s) \sin \pi t.$$

Now let $f : [0, 1]^2 \to \mathbb{R}^4$ be the map whose $i$th component map is $f_i$ for each $i = 1, \ldots, 4$. We claim that $[0, 1]^2/f$ is exactly the Klein bottle. Indeed, $\sim$ is readily checked to be a subset of $\sim_f$. Conversely, suppose $(s, t)$ and $(s', t')$ are two points in $[0, 1]^2$ with $f_i(s, t) = f_i(s', t')$ for each $i$. Squaring the first two of these equations and then summing them up yields $\cos 2\pi s = \cos 2\pi s'$. Then, using those two equations again, we find $\cos 2\pi t = \cos 2\pi t'$ and $\sin 2\pi t = \sin 2\pi t'$, and it follows that either $t = t'$ or $\{t, t'\} = \{0, 1\}$. In the former case, the equations $f_3(s, t) = f_3(s', t')$ and $f_4(s, t) = f_4(s', t')$ yield $\sin 2\pi s = \sin 2\pi s'$, and since $\cos 2\pi s = \cos 2\pi s'$, it follows that either $s = s'$ or $\{s, s'\} = \{0, 1\}$. In the latter case, we get from $f_3(s, t) = f_3(s', t')$ that $\sin 2\pi s = -\sin 2\pi s'$ and since $\cos 2\pi s = \cos 2\pi s'$, we find $s' = 1 - s$.

Conclusion: $\sim_f$ is a subset of $\sim$, and hence $\sim = \sim_f$.

Now we are good to go. It is readily checked that $f$ is continuous and open, and hence, by Proposition 3.1, the Klein bottle $[0, 1]^2/f$ embeds into $\mathbb{R}^4$.

### 3.3 Properties of the Quotient Topology

It is natural to expect that the quotient topology would transfer the properties of the original space to the quotient space, at least when the partition we use for determining the quotient is reasonably well-behaved. Unfortunately, this expectation turns out to be a bit too optimistic. As a case in point, we show that the quotient topology need not inherit the “$T_1$ness” property from the ambient topology of the original space, even when the original space is particularly nice.
Example 3.13. Consider the partition \( \mathbb{R}_{++}, \mathbb{R}_{--}, \{0\} \) of \( \mathbb{R} \), and define the equivalence relation \( \sim \) on \( \mathbb{R} \) by setting \( x \sim y \) iff \( x \) and \( y \) belong to the same member of this partition. Then, it is easily checked that the quotient topology on \( \mathbb{R}/\sim \) is \( \{\mathbb{R}_{++}, \mathbb{R}_{--}, \mathbb{R}/\sim, \{0\}\} \). But then \( \mathbb{R}/\sim \) is not a \( T_1 \)-space, because this topology declares that the only open set that contains \( \{0\} \) is \( \mathbb{R}/\sim \).

Fortunately, there is a nice way of characterizing the quotient topologies that satisfy the \( T_1 \)-condition:

**Proposition 3.2.** Let \( X \) be a topological space, and \( \sim \) an equivalence relation on \( X \). Then, \( X/\sim \) is a \( T_1 \)-space if, and only if, \( [x]_\sim \) is a closed subset of \( X \) for every \( x \in X \).

**Proof.** As we have remarked at the beginning of this section, we have \( C \subseteq C_X \) iff \( \bigcup C \subseteq C_X \), for any subset \( C \) of \( X/\sim \). It follows that \( \{[x]_\sim\} \) is closed in \( X/\sim \) iff \([x]_\sim \) is closed in \( X \), for every \( x \in X \).

Determining when a quotient topology is Hausdorff is a more subtle issue. We often use the following result to deal with this matter.

**Proposition 3.3.** Let \( X \) be a topological space, and \( \sim \) an equivalence relation on \( X \). If \( X/\sim \) is Hausdorff, then \( \sim \) is a closed subset of \( X \times X \). Conversely, if \( \sim \) is a closed subset of \( X \times X \), and \( \pi_\sim \) is an open map, then \( X/\sim \) is Hausdorff.

**Proof.** Define \( f : X \times X \to X/\sim \times X/\sim \) by

\[
f(x, y) := ([x]_\sim, [y]_\sim),
\]

and note that

\[
f(\sim) = \Delta_{X/\sim} \quad \text{and} \quad f^{-1}(\Delta_{X/\sim}) = \sim . \tag{4}
\]

Clearly, \( f \) is continuous, while it is an open map if so is \( \pi_\sim \). Now, if \( X/\sim \) is Hausdorff, then, \( \Delta_{X/\sim} \) is closed (Proposition 1.2), so the second equation in (4) implies that \( \sim \) is closed in \( X \times X \). Conversely, if \( \sim \) is closed, and \( \pi_\sim \) is an open map, then the first equation in (4) implies that the complement of \( \Delta_{X/\sim} \) is open in \( X/\sim \times X/\sim \). By Proposition 1.2, then, \( X/\sim \) is Hausdorff.

The situation is quite pleasant when it comes to separability.

**Proposition 3.4.** Let \( X \) be a topological space, and \( \sim \) an equivalence relation on \( X \). If \( X \) is separable, so is \( X/\sim \).
Proof. This is a consequence of the fact that \( \pi_\sim \) is a continuous surjection from \( X \) onto \( X/\sim \). Indeed, let \( Z \) be a countable dense subset of \( X \). As \( \pi_\sim(Z) \) is obviously countable, we will be done as soon as we show that \( \pi_\sim(Z) \) is dense in \( X/\sim \). To this end, take any nonempty open subset \( O \) of \( X/\sim \), and note that \( Z \cap \pi_\sim^{-1}(O) \subseteq \pi_\sim^{-1}(\pi_\sim(Z) \cap O) \). As \( \pi_\sim \) is a continuous surjection, \( \pi_\sim^{-1}(O) \) is a nonempty open set in \( X \), and hence, \( Z \cap \pi_\sim^{-1}(O) \) is nonempty (because \( Z \) is dense in \( X \)). Thus, \( \pi_\sim^{-1}(\pi_\sim(Z) \cap O) \neq \emptyset \). Conclusion: \( \pi_\sim(Z) \) intersects every nonempty open set in \( X/\sim \).

In general, neither metrizability nor first- (nor second-) countability properties of a topological space is transmitted to a quotient of that space. For example, as you will be asked to show in Exercise 3.19, \( \mathbb{R}/\mathbb{Z} \) is not first-countable. A sufficient condition to obtain a positive result in this regard is given in Exercise 3.20.

**Exercises**

3.1. Show that \( \mathbb{R}/\sim \cong S^1 \), where \( \sim \) is as defined in Example 3.3.

3.2. Prove the claim we made in Example 3.7.

3.3. Consider the equivalence relation \( \sim \) on \( \mathbb{R} \) defined by \( x \sim y \) iff \( x - y \in \mathbb{Q} \). Show that \( \mathbb{R}/\sim \) is an indiscrete space.

3.4. Consider the equivalence relations \( \sim \) and \( \approx \) on \( \mathbb{R}^2 \) defined by \( (x_1, x_2) \sim (y_1, y_2) \) iff \( x_1^2 + x_2^2 = y_1^2 + y_2^2 \), and \( (x_1, x_2) \approx (y_1, y_2) \) iff \( x_1^2 + x_2^2 = y_1^2 + y_2^2 \), respectively. Show that \( \mathbb{R}/\sim \cong \mathbb{R} \) and \( \mathbb{R}/\approx \cong [0, \infty] \). (Hint: View \( \mathbb{R}/\sim \) as the quotient space determined by the map \( (a, b) \mapsto a + b^2 \).)

3.5. Prove: a. \( \mathbb{R}^2/\{0,1\} \cong \mathbb{R}^2 \); b. \( \mathbb{R}^2/\partial \mathbb{D}^2 \cong \mathbb{R}^2 \); c. \( [0,1]^2/\partial [0,1] \cong S^2 \).

3.6. Let \( \sim \) be the smallest equivalence relation on \([0,1]^2 \) such that \( (s,0) \sim (1-s,1) \) and \( (0,t) \sim (1,1-t) \) for each \( s,t \in [0,1] \). Show that \( \mathbb{R}P^2 \cong [0,1]^2/\sim \).

3.7. Let \( \sim \) be the equivalence relation on \( \mathbb{D}^2 \) defined by \( x \sim y \) iff \( x = \pm y \). Show that \( \mathbb{D}^2/\sim \cong \mathbb{D}^2 \).

3.8. The Möbius strip can be characterized as a quotient of the 2-dimensional torus. Indeed, for the equivalence relation \( \sim \) on \( S^1 \times S^1 \) with \( x \sim y \) iff \( y = (x_2, x_1) \), the Möbius strip is homeomorphic to \( (S^1 \times S^1)/\sim \). Prove!

3.9. Let \( X \) be a nonempty set. Let \( d : X \times X \to \mathbb{R}_+ \) be a function that satisfies the Symmetry and Triangle Inequality properties (of being a metric). Assume, in addition, that \( d(x,x) = 0 \) for each \( x \in X \). (Such a map is said to be a semimetric on \( X \).)

a. Show that the binary relation \( \sim \) on \( X \), defined by \( x \sim y \) iff \( d(x,y) = 0 \), is an equivalence relation.
b. Define the map \( D : X \times X \to \mathbb{R}_+ \) by \( D([x]_\sim, [y]_\sim) := d(x, y) \). Show that this is a (well-defined) metric on \( X/\sim \).

c. The topology induced by \( d \) is defined in a way analogous to how the metric topology is defined. (That is, we define the open balls, and hence open sets, in \( X \) relative to the semimetric \( d \) “as if” \( d \) is a metric.) Regarding \( X \) as a topological space relative to this “semimetric topology,” prove that the quotient topology on \( X/\sim \) coincides with the metric topology on \( X/\sim \) induced by \( D \).

3.10. Let \( X \) and \( Y \) be two topological spaces, and consider the equivalence relation on \( X \times Y \) defined by \((x, y) \sim (x, y') \) iff \( y = y' \). Show that \((X \times Y)/\sim \cong X \).

3.11. (Topological Cones) Let \( X \) be a topological space, and let \( \sim \) be the smallest equivalence relation on \( X \times [0, 1] \) such that \((x, 0) \sim (y, 0)\) for every \( x, y \in X \). The quotient space \((X \times [0, 1])/\sim \) is called the cone of \( X \), and is denoted by \( CX \). (Alternatively, we can define \( CX \) as the quotient space \((X \times [0, 1])/X \times \{0\}) \). Intuitively, we may think of \( CX \) as obtained from the cylinder-like space \( X \times [0, 1] \) by bunching the bottom \( X \times \{0\} \) into a single point.

a. Prove that \( C\mathbb{S}^n \cong \mathbb{D}^{n+1} \). (Hint: View \( C\mathbb{S}^n \) as \((\mathbb{S}^n \times [0, 1])/f \), where \( f : \mathbb{S}^n \times [0, 1] \to \mathbb{D}^{n+1} \) is given by \( f(x, t) := tx \).)

b. Let \( X \) be a nonempty closed and bounded subset of \( \mathbb{R}^n \). For any \( x \in \mathbb{R}^n \), let us write \((x, 0) \) for the \((n+1)\)-vector \((x_1, ..., x_n, 0) \) and put \((0, 1) := (0, ..., 0, 1) \in \mathbb{R}^{n+1} \). The following set is sometimes called the geometric cone induced by \( X \):

\[
\Lambda(X) := \{ \lambda(x, 0) + (1 - \lambda)(0, 1) \mid x \in X \text{ and } 0 \leq \lambda \leq 1 \}.
\]

Show that \( CX \cong \Lambda(X) \).

c. Prove that \( CX \) is Hausdorff for any Hausdorff topological space \( X \).

3.12. (Suspensions) Let \( X \) be a topological space, and let \( \sim \) be the smallest equivalence relation on \( X \times [0, 1] \) such that \((x, 0) \sim (y, 0)\) and \((x, 1) \sim (y, 1)\) for every \( x, y \in X \). The quotient space \((X \times [0, 1])/\sim \) is called the suspension of \( X \), and is denoted by \( \Sigma X \). (How would you think about this space geometrically?)

a. Show that \( \Sigma X \cong CX/X \times \{1\} \).

b. Show that \( \Sigma\mathbb{S}^n \cong \mathbb{S}^{n+1} \). (Hint: View \( \Sigma\mathbb{S}^n \) as \((\mathbb{S}^n \times [0, 1])/f \), where \( f : \mathbb{S}^n \times [0, 1] \to \mathbb{S}^{n+1} \) is given by \( f(x, t) := ((\sin \pi t)x_1, (\cos \pi t)x_2) \).

c. Prove that \( CX \) is homeomorphic to a closed subspace of \( \Sigma X \).

3.13. Let \( X \) and \( Y \) be two topological spaces, and \( h : X \to Y \) a homeomorphism. Show that \( X/_{\sim} \cong Y/_{h(S)} \) for any nonempty subset \( S \) of \( X \).
3.14. Let $X$ be a topological space and $S$ a nonempty subset of $X$. Prove: If $S$ is open (closed), then the map $f : X \to X/S$ defined by

$$f(x) = \begin{cases} S, & \text{if } x \in S, \\ \{x\}, & \text{otherwise,} \end{cases}$$

is open (closed). Also show that $X \setminus S \cong (X/S)\setminus \{S\}$, provided that $S$ is either open or closed.

3.15. (*Transitivity of Factorization*) Let $\sim$ be an equivalence relation on a topological space $X$, and $\approx$ an equivalence relation on $X/\sim$. Prove that $(X/\sim)/\approx$ is homeomorphic to $X/\approx$, where $\sim^*$ is the equivalence relation on $X$ given by $x \sim^* y$ iff $[x]_\sim \approx [y]_\approx$.

3.16. Let $X$ be a $T_1$-space and $\sim$ an equivalence relation on $X$.

a. Show that if $\pi_\approx$ is a closed map, then $X/\sim$ is a $T_1$-space.

b. Check that $[0, 1]/\{(1, 2)\}$ is not $T_1$, and use this to conclude that $X/\sim$ may fail to be $T_1$ even when $\pi_\approx$ is an open map.

3.17. (*Smirnov Topology*) Put $S := \{1, \frac{1}{2}, \frac{1}{3}, \ldots\}$, and let $X$ be the set of all real numbers endowed with the following topology:

$$\{O \subseteq T : O \in \mathcal{O}_R \text{ and } T \in 2^S\}.$$

(This topology is usually referred to as either the **Smirnov topology** or the **deleted sequence topology**.)

a. Check that $X$ is Hausdorff, and $S$ is a closed subset of $X$.

b. Consider the equivalence relation $\sim$ on $X$ defined by $x \sim y$ iff either $x = y$ or $\{x, y\} \subseteq S$. Show that $X/\sim$ is a $T_1$-space.

c. Show that $X/\sim$ is not Hausdorff, even though $\sim$ is a closed subset of $X \times X$. (Compare with Proposition 3.3.)

3.18. Put $S := \mathbb{R} \times \{0\}$, and show that $\mathbb{R}^2/\sim$ is not first-countable, even though the quotient map from $\mathbb{R}^2$ onto $\mathbb{R}^2/\sim$ is closed.

3.19. Prove that there does not exist a local basis at $Z$ in $\mathbb{R}/Z$, and conclude that $\mathbb{R}/Z$ is not first-countable.

3.20. Let $X$ be a topological space, and $\sim$ an equivalence relation on $X$. Show that if $X$ is first- (second-) countable, and $\pi^{-1}_\sim(\pi_\sim(O))$ is open in $X$ for every $O \in \mathcal{O}_X$, then $X/\sim$ is first- (second-) countable. (Hint: The image of a first- (second-) countable space under a continuous and open map is first- (second-) countable.)

3.21. (*Identification Maps*) Let $X$ and $W$ be topological spaces, and $p : X \to W$ a surjection. We say that $p$ is an **identification map** if a subset $O$ of $W$ is open in $W$ iff $p^{-1}(O)$ is open in $X$.

a. Show that if $p$ is a homeomorphism, then it is an identification map, and conversely, every injective identification map is a homeomorphism.
b. If \( p \) is continuous, and it is either open or closed, then it is an identification map. But an identification map may be fail to be either open or closed (although it is always continuous). Here is an example. Put \( X := \{(x_1, x_2) \in \mathbb{R}^2 : \text{either } x_1 \geq 0 \text{ or } x_2 = 0\} \), and define \( p : X \to \mathbb{R} \) by \( p(x_1, x_2) := x_1 \). Show that \( p \) is an identification map which is neither open nor closed.

c. Prove: If \( p \) is continuous, and there is a continuous map \( g : W \to X \) such that \( p \circ g = \text{id}_Y \), then \( p \) is an identification map.

d. Show that the (projection) map \( q : X \times W \to X \), defined by \( q(x, y) := x \), is an identification map.

e. Let \( Z \) be a topological space, and \( q : W \to Z \) an identification map. Show that if \( p \) is an identification map, then so is \( q \circ p \).

3.22. (Identification Topology) Let \( X \) be a topological space, \( Z \) a nonempty set, and \( q : X \to Z \) a surjection. The largest topology on \( Z \) such that \( q \) is an identification map is called the identification topology on \( Z \) induced by \( q \).

a. Let \( \sim \) be an equivalence relation on \( X \). Show that the quotient topology on \( X/\sim \) is the identification topology on \( X/\sim \) induced by the map \( \pi_{\sim} \). (So, the quotient topology is a special type of identification topology.)

b. Suppose that \( q \) is an identification map. Let \( Y \) be a topological space, and \( f : X \to Y \) a function which is constant on \( q^{-1}(z) \) for each \( z \in Z \). Prove that there is a unique map \( F : Z \to Y \) such that \( F \circ p = f \). Moreover, \( F \) is continuous iff \( f \) is continuous, and \( F \) is an identification map iff so is \( f \).

c. Deduce Proposition 3.1 from the previous observation.

d. For any \( f : X \to Y \), show that \( F : X/f \to Y \), defined by (3), is a homeomorphism iff \( f \) is an identification map. (Compare with Proposition 3.1.) Conclude that every space \( Y \) with the identification topology induced by a map \( f : X \to Y \) is a quotient space (of the form \( X/\sim_f \)). (So, the identification topology is a special type of quotient topology.)

4 Application: Utility Representation Theory I

4.1 Utility Representation

In economics, a complete preference relation on a set \( X \) of choice alternatives is defined as a total preorder on \( X \), and it is assumed to contain all the information that concerns how an agent compares any two alternatives according to her tastes. However, this preorder is often not the most convenient way of summarizing this information. For instance, maximizing a total preorder is often a much less friendlier exercise than maximizing a real function. Thus, it would be quite useful if we knew how and when one can find a real function that attaches to an alternative \( x \) a (strictly) higher value than an alternative \( y \) iff \( x \) is ranked (strictly) above \( y \) by a given preference relation. Such a function is called the utility function of the individual who possesses this preference relation. A fundamental question in
the theory of individual choice is therefore the following: What sort of complete preference relations can be described by means of a utility function?

To make things precise, let \( \succ \) be a complete preference relation on a nonempty set \( X \). We say that a function \( u : X \to \mathbb{R} \) represents \( \succ \) provided that

\[
x \succ y \quad \text{if and only if} \quad u(x) \geq u(y)
\]

for every \( x \) and \( y \) in \( X \). If such a function exists, we say that \( \succ \) admits a utility representation, and refer to \( u \) as a utility function for \( \succ \).

Let us put on record right away that not every complete preference relation admits a utility representation.

**Example 4.1.** Consider the linear order \( \succ \) on \( \mathbb{R}^2 \) defined as \( x \succ y \) iff \( x_1 > y_1 \) or \( x_1 = y_1 \) and \( x_2 \geq y_2 \). (\( \succ \) is often called the lexicographic order on \( \mathbb{R}^2 \).) This linear order does not admit a utility representation. For, suppose \( u : \mathbb{R}^2 \to \mathbb{R} \) represents \( \succ \). Then, for any real number \( a \), we have \( u(a, a + 1) > u(a, a) \) so that \( I(a) := (u(a, a), u(a, a + 1)) \) is a nonempty interval. Moreover, \( I(a) \cap I(b) = \emptyset \) for any distinct numbers \( a \) and \( b \), because \( u(b, b) > u(a, a + 1) \) whenever \( b > a \), and \( u(b, b + 1) < u(a, a) \) whenever \( b < a \). It follows that the map \( a \mapsto I(a) \) is an injection from \( \mathbb{R} \) into \( \{ I(a) : a \in \mathbb{R} \} \). But since \( \{ I(a) : a \in \mathbb{R} \} \) is countable, this entails that \( \mathbb{R} \) is countable, a contradiction.

On a positive note, it is quite easy to show that every complete preference relation on a nonempty countable set admits a utility representation. Unfortunately, this is not a particularly useful observation, for in most applications the domain of the preference relations in consideration are not countable. (These domains are interpreted as possibly intertemporal commodities, and are often taken as particular convex subsets of normed linear spaces such \( \mathbb{R}^n \) or \( \ell_\infty \).) But with the help of a little bit of topology, we can extend this result to uncountable environments as well.

In what follows, we say that a complete preference relation \( \succeq \) on a topological space \( X \) is upper semicontinuous if \( \{ y \in X : y \succeq x \} \) is a closed subset of \( X \) for every \( x \in X \). We now show that such relations always admit a utility representation, provided that \( X \) has a countable basis.

**Proposition 4.1.** (Rader) Let \( X \) be a second-countable topological space and \( \succeq \) a complete preference relation on \( X \). If \( \succeq \) is upper semicontinuous, then it can be represented by a utility function.

**Proof.** Since \( X \) is second-countable, there is a countable \( \mathcal{O} \subseteq \mathcal{O}_X \) such that \( U = \bigcup \{ O \in \mathcal{O} : O \subseteq U \} \) for every open set \( U \) in \( X \). Enumerate \( \mathcal{O} \) as \( \{ O_1, O_2, \ldots \} \), and put \( \uparrow x := \{ y \in X : x \succ y \} \) for each \( x \) in \( X \). Next, assume that \( \succeq \) is upper semicontinuous, and define \( M(x) := \{ i \in \mathbb{N} : O_i \subseteq \uparrow x \} \) for each \( x \) in \( X \). Then, by transitivity of \( \succeq \), we have \( M(x) \supseteq M(y) \) for every \( x \) and \( y \) in with \( x \succeq y \). Furthermore, because \( \succeq \) is complete and upper semicontinuous, \( \uparrow x \) is an open subset of \( X \), and hence, \( \uparrow x = \bigcup \{ O_i : i \in M(x) \} \) for every \( x \in X \). Thus: \( M(x) \supseteq M(y) \) for
every $x$ and $y$ in $X$ with $x \succ y$. Consequently, the map $u : X \rightarrow [0, 1]$, defined by
\[
u(x) := \sum_{i \in M(x)} 2^{-i},
\]
is a utility function for $\succeq$.

Since endowing a nonempty set with the discrete topology yields a second-countable topological space on which every preference relation is continuous, the following is an immediate consequence of Proposition 4.1.

**Corollary 4.2.** Every complete preference relation on a countable set admits a utility representation.

### 4.2 Digression: Cantor’s Theorem

While it is not topological in nature, there is a fundamental theorem of order theory that relates closely to Proposition 4.1. As we shall use this result later in a topological context, we present it here.

Let $(X, \preceq_X)$ and $(Y, \preceq_Y)$ be two losets. We say that a map $f : X \rightarrow Y$ is order-preserving (where the linear orders on $X$ and $Y$ are implicitly understood) if
\[x \preceq_X y \quad \text{if and only if} \quad f(x) \preceq_Y f(y)
\]
for every $x$ and $y$ in $X$. If this map is bijective, we refer to $f$ as an order-isomorphism from $(X, \preceq_X)$ onto $(Y, \preceq_Y)$. (When there is such a map, we say that $(X, \preceq_X)$ and $(Y, \preceq_Y)$ are order-isomorphic losets.) In this jargon, Corollary 4.2 says that every countable loset is order-isomorphic to a countable subset of $\mathbb{R}$ (where the latter set is ordered by the standard ordering of real numbers). But not all countable subsets of $\mathbb{R}$ have the same order structure. (For instance, $\mathbb{N}$ has a minimum but $\mathbb{Z}$ does not, or between any two rational numbers there is a rational, but this property fails for integers.) An interesting problem is thus to determine all losets that are order-isomorphic to, say, $\mathbb{Q}$. And there is a very nice solution to this problem.

We say that a subset $Z$ of $X$ is $\succeq_X$-dense in $X$ if for every $x$ and $y$ in $X$ with $x \succ_X y$, there is a $z \in Z$ with $x \succ_X z \succ_X y$. For instance, $\mathbb{Q}$ is $\succeq$-dense in $\mathbb{Q}$ (or in $\mathbb{R}$), but $\mathbb{Z}$ is not $\succeq$-dense in $\mathbb{Z}$. It turns out that $\succeq$-denseness of $\mathbb{Q}$, along with this set not possessing a maximum or minimum, characterizes the order structure of $\mathbb{Q}$ completely. (Thus: Any two countable order-dense losets with no endpoints are isomorphic.) This is:

**Cantor’s Theorem.** Let $(X, \succ)$ be a countable loset with no maximum and no minimum elements. If $X$ is $\succeq$-dense (in itself), then $(X, \succ)$ is order-isomorphic to $\mathbb{Q}$.

**Proof.** We shall argue inductively using the following simple fact.

\[
\text{Cantor’s Theorem.}
\]
Claim. Let $A$ be a nonempty finite subset of $X$, and $f : A \to \mathbb{Q}$ an order-preserving map. Then, for any $x \in X \setminus A$, there is an order-preserving map $g : A \cup \{x\} \to \mathbb{Q}$ with $g|_A = f$, and for any $r \in \mathbb{Q} \setminus f(A)$, there are an $x \in X \setminus A$ and an order-preserving map $h : A \cup \{x\} \to \mathbb{Q}$ with $h(x) = r$ and $h|_A = f$.

Proof. Exercise.

Since there is no maximum of $X$ with respect to $\succ$, $X$ must be countably infinite. Let us enumerate $X$ and $\mathbb{Q}$ as $\{x_1, x_2, \ldots\}$ and $\{r_1, r_2, \ldots\}$, respectively. Put $A_1 := \{x_1\}$, and define $f_1 : A_1 \to \mathbb{Q}$ by $f_1(x_1) := r_1$. Now we use the second part of the Claim above to find some $x_k$ in $X$ and an order-preserving $f_2 : A_2 \to \mathbb{Q}$ with $f_2(x_1) = r_1$ and $f_2(x_k) = r_2$, where $A_2 := \{x_1, x_k\}$. Next, let $l$ be the smallest positive integer in $\mathbb{N}\{1, k\}$, put $A_3 := A_2 \cup \{x_l\}$, and use the first part of the Claim above to find an order-preserving map $f_3 : A_3 \to \mathbb{Q}$ with $f_3|_{A_2} = f_2$. Proceeding this way inductively (back and forth) furnishes us a sequence $(A_m)$ of finite subsets of $X$ such that $A_1 \subseteq A_2 \subseteq \cdots$ and $A_1 \cup A_2 \cup \cdots = X$, and a sequence of order-preserving maps $f_m : A_m \to \mathbb{Q}$ such that $f_{m+1}|_{A_m} = f_m$ for each $m \geq 1$ and $f(A_1) \cup f(A_2) \cup \cdots = \mathbb{Q}$. Finally, we define $f := f_1 \cup f_2 \cup \cdots$ (where we view each $f_k$ as a subset of $X \times \mathbb{Q}$). By construction, $f$ is an order-isomorphism from $(X, \succ)$ onto $\mathbb{Q}$.

Here is a nice application of this theorem (which we shall use in Chapter 7).

Example 4.2. Let $L$ stand for the set of all left endpoints of all open intervals deleted in the construction of the Cantor set, and let $R$ stand for the set of all right endpoints of those intervals. (Recall Exercise 4.7 of Chapter 1.) An easy application of Cantor’s Theorem shows that both $L$ and $R$ are order-isomorphic to $\mathbb{Q}$. It follows that endowing $\mathbb{Q} \times \{0, 1\}$ with the lexicographic order yields a lost that is order-isomorphic to $L \cup R$ (where the latter set is ordered by the usual ordering of the reals).

4.3 Semicontinuous Utility Representation

It turns out that Proposition 4.1 is not a first-best result. Under its hypotheses, we can not only find a utility function for the given preference relation, but we can in fact choose such a function to be upper semicontinuous (Exercise 4.5 of Chapter 4). We now prove this fact by using some of what we have learned about quotient topology in Section 3.

Rader’s Utility Representation Theorem. Let $X$ be a second-countable topological space and $\succsim$ a complete preference relation on $X$. If $\succsim$ is upper semicontinuous, then it can be represented by an upper semicontinuous utility function.

Proof. (Richter) We apply Proposition 4.1 to obtain a utility function $u$ for $\succsim$. Without loss of generality, we assume that $u$ is bounded. (Otherwise, replace $u$ with, say, $\arctan u$; see Exercise 4.1.) Next, we note that the symmetric part $\sim$ of $\succsim$ is an equivalence relation on $X$, and endow $X/\sim$ with the quotient topology.
Define the map $U : X/\sim \to \mathbb{R}$ by $U([x]_\sim) := u(x)$. Since $u$ represents $\succeq$, it assigns the same value to all members of any given $[x]_\sim$. Therefore, $U$ is well-defined. Let $V$ denote the limsup of $U$ (relative to the quotient topology), that is, define $V : X/\sim \to \mathbb{R}$ by

$$V([x]_\sim) := \inf_{O \in \mathcal{O}_\sim(x)} \left( \sup_{[\omega]_\sim \in O} u(\omega) \right),$$

where $\mathcal{O}_\sim(x)$ is the collection of all open subsets of $X/\sim$ that contain $[x]_\sim$. As the limsup of any bounded real function on a topological space is upper semicontinuous (Exercise 4.5 of Chapter 4), $V$ is an upper semicontinuous function on $X/\sim$. Since $\pi_\sim$ is continuous, it follows that $v := V \circ \pi_\sim$ is an upper semicontinuous real map on $X$. For future reference, we note that

$$v(x) \geq u(x) \quad \text{for every } x \in X, \quad (5)$$

which is an obvious consequence of the definitions of $V$ and $v$.

We will complete our proof by showing that $v$ is a utility function for $\succeq$. To this end, take any elements $x$ and $y$ of $X$. If $x \sim y$, then, clearly, $v(x) = V([x]_\sim) = V([y]_\sim) = v(y)$. Suppose, then, $x \succ y$. We wish to show that $v(x) > v(y)$ by distinguishing between two cases.

**Case 1**: $x \succ z \succ y$ for some $z \in X$. In this case, the set $O := \{[\omega]_\sim : z \succ \omega\}$ includes $[y]_\sim$, and we have

$$u(z) \geq \sup_{[\omega]_\sim \in O} u(\omega).$$

Furthermore, by upper semicontinuity and completeness of $\succeq$, the set $\{z\}$ is open in $X$, while $\{z\} = \bigcup\{[\omega]_\sim : [\omega]_\sim \in O\}$. Thus, by definition of the quotient topology, $O$ is an open subset of $X/\sim$, that is, $O \in \mathcal{O}_\sim(y)$. It then follows from the definition of $v$ that $u(z) \geq V([y]_\sim) = v(y)$. By (5), then, $v(x) \geq u(x) > u(z) \geq v(y)$ since $u$ represents $\succeq$.

**Case 2**: $x \succ z \succ y$ for no $z \in X$. In this case,

$$u(y) = \max_{[\omega]_\sim \in O} u(\omega)$$

where $O := \{[\omega]_\sim : x \succ \omega\}$. As, again, $O \in \mathcal{O}_\sim(y)$, it follows from the definition of $v$ that $u(y) \geq v(y)$. By (5), then, $u(y) = v(y)$, and hence, $v(x) \geq u(x) > u(y) = v(y)$ since $u$ represents $\succeq$.

Curiously, we cannot furnish a continuous utility function from the assumptions of Rader’s Theorem. (In fact, there does not have to be a continuous utility function for an upper semicontinuous preference relation even on a Euclidean space.) We will return to the problem of finding continuous utility functions for a complete preference relation in the next chapter.

**Exercises**
4.1. (Uniqueness of Utility Representations) Let $\succeq$ be a complete preference relation on a nonempty set $X$, and $u$ and $v$ two real functions on $X$. Prove: $u$ and $v$ are both utility functions for $\succeq$ iff there exists a strictly increasing real function $f$ on $u(X)$ such that $v = f \circ u$.

4.2. (Wold) Let $\succeq$ be a complete preference relation on $\mathbb{R}_+^n$ such that both \( \{ y \in X : y \succeq x \} \) and \( \{ y \in X : x \succeq y \} \) are closed subsets of $X$ for every $x \in X$. Assume that $\succeq$ is monotonic in the sense that $x \succ y$ holds for every distinct $x$ and $y$ in $\mathbb{R}_+^n$ with $x \geq y$. Prove that the map $x \mapsto \max\{a \geq 0 : x \succeq a1\}$, where $1$ is the $n$-vector of $1$s, is a continuous utility function for $\succeq$.

4.3. Give an example of an upper semicontinuous preference relation on $\mathbb{R}^2$ that does not admit a continuous utility representation.

4.4. (A Generalization of Rader’s Utility Representation Theorem) Let $X$ be a nonempty set, and $\succeq$ a complete preference relation on $X$. Assume that there is a countable subset $Z$ of $X$ such that for every $x$ and $y$ in $X$ with $x \succ y$, we have $x \succeq z \succ w \succeq y$ for some $z, w \in Z$. First, show that $\succeq$ admits a utility function that is upper semicontinuous with respect to the topology on $X$ generated by using \( \{ z : z \in Z \} \) as a subbasis. (Here $\{ z : z \in Z \} \) And then derive Rader’s Representation Theorem from this observation.