Chapter 8
Applications to Dynamical Systems
Topological Dynamics

1 Discrete-Time Dynamical Systems

In this chapter we take a bit of a break from general topology, and turn instead to a major application. Our task is to make a modest, yet topological, introduction to the vast field of (discrete-time) dynamical systems. If anything, this will show how far reaching general topology is insofar as its usage in other fields go. Indeed, throughout the exposition, we will use parts of virtually everything covered so far in the text. In later chapters, as we learn more about topology, we will revisit the work of this chapter, and add more to it.

1.1 Introduction

First Impressions

Many types of intertemporal phenomena are modeled through first-order difference equations of the type

$$x_{m+1} = f_m(x_m), \quad m = 0, 1, ...$$

(1)

Here $x_0$ stands for the initial value of some variable of interest (such as price of a certain financial asset or portfolio, or size of some other characteristic) of a population, or the place of rest of a physical entity before it is impacted by some force). In deterministic models, this initial value is assumed to be known, and the value $x_m$ of the variable in question at time $m$ is determined recursively through the equation (1). Here, for any nonnegative integer $m$, $f_m$ is a given self-map on a nonempty set $X$. The set $X$ is called the state
space (or phase space) of the system, and it describes all values that our variable may take. In turn, \( f_m \) tells us how the value of our economic variable at time \( m+1 \) depends on its value at the previous period. This model, which is an example of a discrete dynamical system, is said to be time-dependent because the nature of transitions of our variable may change from period to period. (For, \( f_1, f_2, \ldots \) may be different functions.)

Clearly, conditional on \( x_0 \), the unique solution to the difference equation (1) is given by the sequence \( (f_m \circ \cdots \circ f_1 (x_0)) \) in \( X \). Unfortunately, our ability of “solving” (1) in this manner is of little use in practice, because the qualitative nature of this solution sequence is impossible to understand except in the case of very special examples. Especially when the given system is of abstract nature, we need radically new approaches to deal with questions such as “Will we ever come back to a state near the original state \( x_0 \)” or “If \( x'_0 \) is a state that is near \( x_0 \), will the solutions emanating from \( x_0 \) and \( x'_0 \) remain close to each other eventually as well?” and so on.

There are two main approaches toward studying the dynamics of a difference equation such as (1). The first models the state space \( X \) as a measure space, and the transition maps \( f_m \) as certain types of measurable transformations; this theory is often referred to as measurable dynamics and ergodic theory. The second models the state space as a topological space and \( f_m \) as continuous maps. This theory is called topological dynamics, and is the subject matter of the present chapter.

First, a simplification. All dynamical systems we will consider in this chapter are time-independent, that is, the structure of their transitions remain independent of time. Formally, this means that we look at the difference equation (1) under the assumption that \( f_1 = f_2 = \cdots \), that is, in our case, the dynamical system at hand is described by means of the first-order difference equation

\[
x_{m+1} = f(x_m), \quad m = 0, 1, \ldots,
\]

where \( f \) is a given self-map on the state space \( X \) that dictates the transition of our variable through time. This leads us to the following fundamental definition.

**Definition.** A discrete-time (topological) dynamical system is an ordered pair \((X, f)\) where \( X \) is a Hausdorff space and \( f \) a continuous self-map on \( X \). For brevity, we will refer to any such system simply as a dynamical system throughout the exposition, and refer to \( f \) as its transition function.
The set of all states that the trajectory of \( x \) under \( f \) visits is called the orbit of \( x \) under \( f \), and is denoted as \( \text{orb}_f(x) \), that is,

\[
\text{orb}_f(x) := \{x, f(x), f^2(x), \ldots\}.
\]

If \( x \) is a fixed point of \( f \), then, and only then, this set is a singleton: \( \text{orb}_f(x) = \{ x \} \). More generally, for any positive integer \( k \), if \( x \) is a fixed point of \( f^k \), that is, \( f^k(x) = x \), then this set has at most \( k \) elements: \( \text{orb}_f(x) \subseteq \{ x, \ldots, f^{k-1}(x) \} \). In this case, we say that \( x \) is periodic under \( f \), and refer to the smallest \( k \in \mathbb{N} \) such that \( f^k(x) = x \) as the period of \( x \) under \( f \). (Clearly, \( k \) is the period of \( x \) under \( k \) iff \( \text{orb}_f(x) = \{ x, \ldots, f^{k-1}(x) \} \} \).

The set of all points in \( X \) that are periodic under \( f \) is denoted by \( \text{Per}(X, f) \), that is,

\[
\text{Per}(X, f) := \bigcup_{k=1}^{\infty} \{ x \in X : x \text{ is fixed point of } f^k \}.
\]

Finally, we say that \( x \) is eventually periodic under \( f \) if \( f^m(x) \in \text{Per}(X, f) \) for some nonnegative integer \( m \). In other words, \( x \) is eventually periodic under \( f \) iff \( \text{orb}_f(x) \) is a finite set. (See Figure 8.1 for an abstract illustration of these concepts.)

1.2 Examples of Dynamical Systems

In this section, we provide a small bouquet of concrete dynamical systems. We will return to these examples as we develop the theory throughout the
Example 1.1. \((X, \text{id}_X)\) is a dynamical system for any Hausdorff space \(X\). This system is trivial in that it precludes any dynamics; wherever the system starts, it never moves. For, obviously, the trajectory of any \(x \in X\) under \(\text{id}_X\) is \((x, x, \ldots)\). Obviously, \(\text{Per}(X, \text{id}_X) = X\).

Example 1.2. (Permutations) Let \(n\) be an integer with \(n \geq 2\), and consider \(X := \{x_1, \ldots, x_n\}\) as a discrete space. Then, \((X, f)\) is a dynamical system for any bijective self-map \(f\) on \(X\). (In this context, \(f\) is called a permutation on \(X\).) As \(X\) is finite, we clearly have \(\text{Per}(X, f) = X\) here.

For any nonempty \(S \subseteq X\) with \(k := |S| \geq 2\), say, \(S = \{x_i, \ldots, x_k\}\), we say that \(f\) is an \(S\)-cycle if \(f(x_j) = x_{j+1 \text{ (mod } k)}\) for each \(j \in \{1, \ldots, k\}\), and \(f(x) = x\) for each \(x \in X\setminus S\). For such an \(f\), the period of every \(x \in S\) under \(f\) is \(k\) while the period of every \(x \in X\setminus S\) is 1. An \(X\)-cycle is said to be an irreducible permutation on \(X\). For any such \(f\), \(\text{orb}_f(x) = X\), and hence the period of \(x\) under \(f\) is \(n\), for every \(x \in X\).

Example 1.3. Let \(X\) be a complete metric space and \(f\) a contraction on \(X\). Clearly, \((X, f)\) is a dynamical system. Moreover, by the (proof of) the Banach Fixed Point Theorem, there exists a (unique) state \(x^*\) in \(X\) such that \(f^n(x) \to x^*\) for every \(x \in X\). Thus, no state in \(X\setminus\{x^*\}\) is periodic under \(f\), while the period of \(x^*\) under \(f\) is 1. Thus: \(\text{Per}(X, f) = \{x^*\}\).

Example 1.4. (Translations on \([0,1]\)) Define \(d : [0,1] \times [0,1] \to \mathbb{R}_+\) by
\[
d(x, y) := \min\{|x - y|, 1 - |x - y|\},
\]
which is readily checked to be a metric on \([0,1]\). We call \(d\) the Kronecker metric, and denote by \(X\) the metric space that results by endowing \([0,1]\) with this metric. Note that the Kronecker metric is not equivalent to the absolute value metric on \([0,1]\), that is, \(X\) is not homeomorphic to \([0,1]\). In fact, \(X\) is a compact metric space.

For any real number \(\alpha \in [0,1]\), the \(\alpha\)-translation on \(X\) is the self-map \(f_\alpha\) on \(X\) defined by \(f_\alpha(x) := x + \alpha - \lfloor x + \alpha \rfloor\), or put more explicitly,
\[
f_\alpha(x) := \begin{cases} 
  x + \alpha, & \text{if } x + \alpha < 1 \\
  x + \alpha - 1, & \text{otherwise}.
\end{cases}
\]
(See Figure 8.2.) It is an easy exercise to check that \(f_\alpha\) is a continuous self-map on \(X\) (while, of course, \(f_\alpha\) is not continuous on \([0,1]\) relative to the absolute metric unless \(\alpha = 0\)). In fact,
\[
d(f_\alpha(x), f_\alpha(y)) = d(x, y) \quad \text{for every } x, y, \alpha \in [0,1],
\]
so \( f_\alpha \) is actually an isometry from \( X \) onto \( X \).

Now, we know that \((X, f_\alpha)\) is a dynamical system for every \( \alpha \in [0, 1) \).
(If \( \alpha = 0 \), this system is a special case of the trivial system we considered in Example 1.1.) Let us look at the structure of the orbit of an arbitrarily fixed point \( x \) in \( X \) under \( f_\alpha \). To this end, we note that

\[
 f_\alpha^m(x) = x + m\alpha - \lfloor x + m\alpha \rfloor, \quad m = 1, 2, \ldots
\]

for any \( \alpha \in [0, 1) \), which is an easy exercise to check. Thus, if \( \alpha > 0 \) is rational, say, \( \alpha = n/m \) for some relatively prime positive integers \( m \) and \( n \) with \( n < m \), we have \( f_\alpha^m = \text{id}_{[0, 1)} \), and hence orb\(_{f_\alpha}(x) = \{x, f_\alpha(x), \ldots, f_\alpha^{m-1}(x)\} \) for any \( x \in [0, 1) \). Conclusion: If \( \alpha \) is a rational number in \([0, 1)\), then every state in \( X \) is periodic under \( f_\alpha \). (In fact, all \( x \) in \([0, 1)\) has the same period under \( f_\alpha \) in this case.)

Now suppose \( \alpha \) is an irrational number. This changes the situation dramatically. First, note that \( f_\alpha^m(x) \neq f_\alpha^n(x) \) for any distinct \( m, n \in \mathbb{Z}_+ \). Indeed, in view of (4), \( f_\alpha^m(x) = f_\alpha^n(x) \) implies that \((m - n)\alpha\) is the difference of two integers, which is possible iff \( m = n \) (because \( \alpha \) is irrational). Next, take an arbitrary \( \varepsilon \in (0, \frac{1}{2}) \). By the Bolzano-Weierstrass Theorem, there are nonnegative integers \( m \) and \( n \) such that \( m > n \) and \( |f_\alpha^m(x) - f_\alpha^n(x)| < \varepsilon \). Then, applying (3) inductively,

\[
 d(f_\alpha^{m-n}(x), x) = d(f_\alpha^m(x), f_\alpha^n(x)) < \varepsilon.
\]

So, all terms of the sequence \((x, f_\alpha^{m-n}(x), f_\alpha^{2(m-n)}(x), \ldots)\) are distinct, any two consecutive terms of it are a fixed distance (in terms of the Kronecker metric) away from each other (by (3)), and that distance is less than \( \varepsilon \). Therefore, any number in \([0, 1)\) is at most \( \varepsilon/2 \) away from at least one term of this sequence (in terms of the Kronecker metric), and hence of an element of orb\(_{f_\alpha}(x) \). In view of the arbitrariness of \( \varepsilon \), therefore, we may conclude that orb\(_{f_\alpha}(x) \) is dense in \( X \). Conclusion:

\[
 \text{Per}(X, f_\alpha) = \begin{cases} 
 X, & \text{if } \alpha \text{ is rational}, \\
 \emptyset, & \text{if } \alpha \text{ is irrational}.
\end{cases}
\]

**Example 1.5.** (The Doubling Map) Let \( X \) be as in the previous example, and define \( f : X \to X \) by \( f(x) := 2x \mod 1 \). (This map is called the **doubling map**; see Figure 8.3.) Then, \((X, f)\) is a dynamical system. Obviously, 0 is

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\(^1\)The fact that orb\(_{f_\alpha}(x) \) is finite when \( \alpha \) is rational, and dense in \( X \) when \( \alpha \) is irrational is known as Kronecker’s Theorem.
the only fixed point of \( f \). On the other hand, for any positive integer \( k \geq 2 \) and \( x \in X \setminus \{0\} \), we have \( f^k(x) = x \) iff \( 2^k x \equiv x \pmod{1} \) which holds iff \( 2^k x = x + l \) for some positive integer \( l \), that is, \( x = l/(2^k - 1) \). Thus, the periods of any one of the \( 2^k - 2 \) points \( \frac{1}{2^k-1}, \ldots, \frac{2^k-2}{2^k-1} \), and only of these points, equal \( k \). (Note. It follows fairly easily from this observation that \( \text{Per}(X, f) \) is dense in \( X \).) As we shall see later, there are also points in \( X \) that have dense orbits.

An element \( x \) of \( X \) is eventually periodic under \( f \) iff it is rational. Indeed, if \( x \) is eventually periodic, then there exist \( (m, k) \in \mathbb{Z}_+ \times \mathbb{N} \) with \( f^m(x) = f^{m+k}(x) \), that is, \( 2^m x - [2^m x] = 2^{m+k} x - [2^{m+k} x] \), so \( (2^{m+k} - 2^m) x \in \mathbb{Z}_+ \), and this implies \( x \in \mathbb{Q} \). Conversely, if \( x \) is rational, then \( x = m/n \) for some \( (m, n) \in \mathbb{Z}_+ \times \mathbb{N} \) with \( m < n \), and it follows that \( \{2^k \frac{m}{n} - \lfloor 2^k \frac{m}{n} \rfloor : k \in \mathbb{N}\} \) is a finite set (because there are only finitely many nonnegative fractions of the form \( m/n \) with \( m < n \)).

**Example 1.6. (Shifts)** Let \( \Omega \) be a Hausdorff space, and put \( X := \Omega^\infty \) which we think of as a topological space relative to the product topology. Consider the map \( \sigma : X \to X \) defined by

\[
\sigma(x_1, x_2, \ldots) := (x_2, x_3, \ldots).
\]

This map, which is called the (one-sided) **shift operator** on \( X \), is readily checked to be continuous, so \( (X, \sigma) \) is a dynamical system.

For any positive integer \( k \) and \( x \in X \), the equation \( \sigma^k(x) = x \) means that \( (x_{k+1}, x_{k+2}, \ldots) = (x_1, x_2, \ldots) \), and hence

\[
(x_1, x_2, \ldots) = (x_{k+1}, x_{k+2}, \ldots) = (x_{2k+1}, x_{2k+2}, \ldots) = \cdots
\]

It follows that the period of a point \( x \) in \( X \) under \( \sigma \) is \( k \) iff the first \( k \) terms of the sequence \( x \) repeats themselves as a block indefinitely. An easy consequence of this observation is that \( \text{Per}(X, \sigma) \) is dense in \( X \).

Dynamical systems of the form considered in Example 1.6 play a major role in the general theory of dynamical systems. In most cases of interest, however, \( \Omega \) is taken as a finite set (of symbols) that is viewed as a discrete metric space. (In that case, the resulting system \((X, f)\) is called a **symbolic dynamical system**.) We work with this special case in our next example.

**Example 1.7. (Topological Markov Chains)** Let \( n \) be a positive integer with \( n \geq 2 \), and put \( \Omega := \{\omega_1, \ldots, \omega_n\} \). Let \( \longrightarrow \) stand for a (nonempty) binary relation on \( \Omega \), but let us agree to write \( \omega_i \longrightarrow \omega_j \) if \( (\omega_i, \omega_j) \) belongs to \( \longrightarrow \).
In addition, we assume that for any \( k = 1, \ldots, n \), if \( \omega_k \to \omega_j \) does not hold for any \( j \), then \( \omega_i \to \omega_k \) does not hold for any \( i \). As for an interpretation, we may think of \( \Omega \) here as an “alphabet,” and consider any sequence in \( X := \Omega^\infty \) as a “word.” Then, as a rule of grammar, the binary relation \( \to \) tells us which letters of the alphabet can possibly come right after any one given letter. Consequently, the set of all grammatically admissible words depends on \( \to \), and is given by

\[
X_\to := \{ (\omega_{i_1}, \omega_{i_2}, \ldots) \in X : \omega_{i_k} \to \omega_{i_{k+1}} \text{ for every } k = 1, 2, \ldots \}.
\]

Thanks to the nonemptiness of \( \to \) and the assumption we made on \( \to \), we have \( X_\to \neq \emptyset \).

This story is actually better told if we use the matrix representation of the relation \( \to \). To this end, define \( A \) to be the \( n \times n \) matrix whose \( ij \)th entry \( a_{ij} \) equals \( 1 \) if \( \omega_i \to \omega_j \) and \( 0 \) otherwise. (The assumptions we made on \( \to \) ensure that at least one entry of \( A \) is \( 1 \), and that if the \( j \)th row of \( A \) consists of all \( 0 \)s, then so does its \( j \)th column.) Given this representation, we denote \( X_\to \) as \( X_A \), that is,

\[
X_A := \{ (\omega_{i_1}, \omega_{i_2}, \ldots) \in X : a_{i_k,i_{k+1}} = 1 \text{ for every } k = 1, 2, \ldots \}.
\]

We think of this set as a metric subspace of \( X \) (that is, endow it with the product topology). Then, where \( \sigma \) is the shift operator on \( X \), we have \( \sigma(X_A) \subseteq X_A \), and hence \( (X_A, \sigma|_{X_A}) \) is a dynamical system, which is called the **topological Markov chain** induced by \( A \). This system is particularly interesting when \( A \) is **irreducible**, that is, when there is a positive integer \( M \) such that all entries of \( A^M \) are positive, where \( A^1 := A \), \( A^2 := AA \), \( A^3 := AA^2 \), and so on. You will be asked below to prove that \( \operatorname{Per}(X_A, \sigma|_{X_A}) \) is dense in \( X_A \), provided that \( A \) is irreducible.

**Warning.** Topological Markov chains were introduced by William Parry in 1964 who called them **subshifts of finite type**. For this reason, the dynamical system \( (X_A, \sigma|_{X_A}) \) is sometimes referred to as such in the literature.

**Dynamical Subsystems**

Let \( (X, f) \) be a dynamical system. A nonempty subset \( S \) of \( X \) is said to be **\( f \)-invariant** if \( f(S) \subseteq S \) and **strongly \( f \)-invariant** (or a **fixed set** of \( f \)) if \( f(S) = S \). Invariant sets are important because they allow us to obtain new dynamical systems within a given system. To wit, for any subspace \( Y \) of \( X \), \( (Y, f|_Y) \) is a dynamical system iff \( Y \) is \( f \)-invariant.
Example 1.8. Let \((X, f)\) be a dynamical system, and pick any state \(x\) in \(X\). Then,
\[
f(\text{orb}_f(x)) = \{f(x), f^2(x), \ldots\} \subseteq \{x, f(x), \ldots\} = \text{orb}_f(x).
\]
Thus: The orbit of any point in a dynamical system is invariant (but not necessarily strongly invariant). Besides,
\[
f(\text{cl}(\text{orb}_f(x))) = \text{cl}(f(\text{orb}_f(x))) \subseteq \text{cl}(\text{orb}_f(x)).
\]
Thus: The closure of the orbit of any point in a dynamical system is a closed invariant set (but it is not necessarily strongly invariant). Consequently,
\[
\left(\text{orb}_f(x), f|_{\text{orb}_f(x)}\right) \quad \text{and} \quad \left(\text{cl}(\text{orb}_f(x)), f|_{\text{cl}(\text{orb}_f(x))}\right)
\]
are both dynamical systems.

Unlike invariant sets, strongly invariant sets are often difficult to come by. (Example. There is no strongly \(f\)-invariant subset of \(\mathbb{R}_{++}\), where \(f\) is the self-map on \(\mathbb{R}_{++}\) with \(f(x) := 2x\).) But we can always find strongly invariant sets for dynamical systems that have compact state spaces. (Recall Exercise.) More generally:

Lemma 1.1. Let \((X, f)\) be a dynamical system, and suppose \(T\) is a compact \(f\)-invariant subset of \(X\). Then, there is a compact and strongly \(f\)-invariant subset of \(T\).

Proof. Being a continuous image of a compact set, \(f^k(T)\) is a compact set in \(X\) for every positive integer. Moreover, since \(T\) is \(f\)-invariant, \(T \supseteq f(T) \supseteq f^2(T) \supseteq \cdots\), so we may apply Proposition 1.1 of Chapter 7 to conclude that \(S := \bigcap_{i=1}^{\infty} f^i(T)\) is a nonempty compact subset of \(T\). It is readily checked that \(f(S) \subseteq S\). To show that the converse containment holds as well, take any \(x \in S\), which means that \(x \in f^k(T)\) for every \(k \in \mathbb{N}\). Then, for any \(k = 2, 3, \ldots,\) put \(A_k := f^{-1}(x) \cap f^{k-1}(T)\), and notice that this is a nonempty closed subset of \(T\). (Nonemptiness of \(A_k\) follows from the fact that \(x \in f^k(T)\) implies \(f(y) = x\) for some \(y \in f^{k-1}(T)\), which means \(y \in A_k\).) As \(A_1 \supseteq A_2 \supseteq \cdots\), applying Proposition 1.1 of Chapter 7 one more time yields \(\bigcap_{i=2}^{\infty} A_i \neq \emptyset\). Let us then pick any \(z\) in this set. Then, \(f(z) = x\) and \(z \in f^{k-1}(T)\) for each \(k \geq 2\). But the latter statement means that \(z \in S\), so we find \(x \in f(S)\), as we sought.

We will use this fact in Section 2.
1.3 Topological Conjugacy

The following definition specifies in what way we can think of two dynamical systems as “equivalent.”

**Definition.** Let \((X, f)\) and \((Y, g)\) be two dynamical systems. A map \(\varphi : X \to Y\) is said to be a **topological conjugacy** between \((X, f)\) and \((Y, g)\) if \(\varphi\) is a homeomorphism and \(\varphi \circ f = g \circ \varphi\). If there exists such a map, we say that \((X, f)\) and \((Y, g)\) are **topologically conjugate**, and write \((X, f) \cong (Y, g)\).

**Warning.** This notation is consistent with how we have so far used the symbol \(\cong\). Indeed, \(X \cong Y\) if \((X, \text{id}_X) \cong (Y, \text{id}_Y)\).

If \((X, f)\) and \((Y, g)\) are topologically conjugate dynamical systems, then not only the topologies of \(X\) and \(Y\) are indistinguishable from each other, but the trajectories of points under the maps \(f\) and \(g\) are equivalent up to relabeling. To wit, let \(\varphi\) be a topological conjugacy between \((X, f)\) and \((Y, g)\), take any point \(x_0\) in \(X\), and define \(x_{m+1} := f(x_m)\) for each \(m = 0, 1, \ldots\) Clearly, \(\varphi\) pairs \(x_0\) with the point \(y_0 := \varphi(x_0)\) in \(Y\). Put \(y_{m+1} := g(y_m)\) for each \(m = 0, 1, \ldots\) The point here is that pairing of \(x_0\) and \(y_0\) by \(\varphi\) extends to the pairing \(x_m\) and \(y_m\) for any \(m \geq 0\). Indeed, \(y_1 = g(\varphi(x_0)) = \varphi(f(x_0)) = \varphi(x_1)\). In turn, \(y_2 = g(y_1) = g(\varphi(x_1)) = \varphi(f(x_1)) = \varphi(x_2)\), and continuing inductively, we find that \(y_m = \varphi(x_m)\) for every \(m \geq 0\). In other words, evaluating \(\varphi\) at the trajectory of \(x_0\) under \(f\) term by term yields precisely the trajectory of \(\varphi(x_0)\) under \(g\). (In particular, \(\varphi(\text{orb}_f(x_0)) = \text{orb}_g(\varphi(x_0))\).) Consequently, for any \(x_0 \in X\), the topological structure of the trajectory of \(x_0\) under \(f\) and that of \(\varphi(x_0)\) under \(g\), and hence the dynamics induced by the maps \(f\) and \(g\), are bound to be the same. For instance, the number of fixed points of \(f\) and \(g\) are the same, and a point \(x\) is (almost) periodic under \(f\) iff \(\varphi(x_0)\) is (almost) periodic under \(g\).

**Examples**

Here are some examples of topologically conjugate dynamical systems.

**Example 1.9.** (Rotations and Translations) Take any \(\alpha \in [0, 1)\), and consider the dynamical system \((\mathbb{S}^1, g_\alpha)\) where \(g_\alpha(\cos 2\pi \theta, \sin 2\pi \theta) := (\cos 2\pi (\theta + \alpha), \sin 2\pi (\theta + \alpha))\) for every \(\theta \in [0, 1)\). (Geometrically speaking, \(g_\alpha\) rotates any given point on the unit circle \(\mathbb{S}^1\) counter clockwise by the angle \(2\pi \alpha\).) Then, \((X, f_\alpha) \cong (\mathbb{S}^1, g_\alpha)\), where \((X, f_\alpha)\) is the dynamical system we have introduced in Example 1.4. Indeed, \(\varphi : X \to \mathbb{S}^1\), defined by

\[
\varphi(x) := (\cos 2\pi x, \sin 2\pi x),
\]

9
is a homeomorphism. (Recall that the topology of $X$ is the Kronecker metric topology introduced in Example 1.4, not the usual one.) Moreover, for any point $x$ in $X$, the sequence $(\varphi(x), \varphi(f_{\alpha}(x)), \varphi(f_{\alpha}^2(x)), \ldots)$ is the same as the trajectory of $\varphi(x)$ under $g_{\alpha}$. (This is illustrated in Figure 8.4 in the case where $\alpha = \frac{1}{4}$ and $x = \frac{3}{10}$. ) Indeed, for any $x \in X$,

$$
\varphi(f_{\alpha}(x)) = \varphi(x + \alpha - \lfloor x + \alpha \rfloor) = (\cos 2\pi(x + \alpha), \sin 2\pi(x + \alpha)) = g_{\alpha}(\cos 2\pi x, \sin 2\pi x) = g_{\alpha}(\varphi(x)).
$$

It is in this sense that translations on $[0, 1)$ and rotations on $S^1$ describe equivalent dynamics.

**Example 1.10.** Let $C$ stand for the Cantor set and define $f : C \to [0, 1]$ by

$$
f(x) := \begin{cases} 
3x, & \text{if } x \in [0, \frac{1}{3}] \cap C \\
3x - 2, & \text{if } x \in [\frac{2}{3}, 1] \cap C.
\end{cases}
$$

Then, $f$ is continuous (by the Pasting Lemma), and one can easily verify that $f(C) \subseteq C$. Thus, $(C, f)$ is a dynamical system. (Some authors refer to $(C, f)$ as the Cantor system.) Now consider the dynamical system $((0, 1)\infty, \sigma)$ where $\sigma$ is the shift operator on $\{0, 1\}\infty$. These two systems are topologically conjugate: $((0, 1)\infty, \sigma) \cong (C, f)$. Indeed, $\varphi : \{0, 1\}\infty \to C$, defined by

$$
\varphi(x_0, x_1, \ldots) := \sum_{i=0}^{\infty} \frac{2x_i}{3^{i+1}},
$$

is a homeomorphism. (As we have noted in the previous chapter, this is why $\{0, 1\}\infty$ is called the Cantor space.) We leave it as an exercise to verify that $\varphi \circ \sigma = f \circ \varphi$.

**A Major Example: Iteration of Quadratic Maps on the Line**

The following example is a bit more far reaching than the previous two, and demonstrates how amazingly useful symbolic dynamical systems are.

**Example 1.11.** Fix a real number $\lambda > 2 + \sqrt{5}$ (a restriction to be demystified shortly), and consider the self-map $f$ on $\mathbb{R}$ defined by

$$
f(x) := \lambda x(1 - x).
$$
It is readily checked that $x$ escapes to $-\infty$ under $f$, that is, $f^m(x) \to -\infty$, when $x \in (-\infty,0)$ or $x \in (1,\infty)$. Thus, states whose orbits under $f$ are bounded belong to $[0,1]$. But not all states in $[0,1]$ have such orbits. If, for instance, $x \in [0,1]$ is such that $f(x) > 1$, then $f^m(f(x)) \to -\infty$, so $x$ again escapes to $-\infty$ under $f$. And, given the parametric restriction we impose on $\lambda$, there are such points. In particular, solving the quadratic equation $f(x) = 1$ shows that $f(x) \in [0,1]$ holds only for $x \in [0,\alpha] \cup [1 - \alpha,1]$, where $\alpha := \frac{1}{2} - \sqrt{\frac{1}{4} - \frac{1}{\lambda}}$. (See Figure 8.5.) Iterating this reasoning, we see that an $x \in \mathbb{R}$ has a bounded orbit under $f$ if and only if $0 \leq f^i(x) \leq 1$ for every nonnegative integer $i$. Thus, where $\Lambda$ stands for the collection of all real numbers $x$ such that $\text{orb}_f(x)$ is bounded, we have

$$\Lambda = \bigcap_{i=0}^{\infty} f^{-i}([0,1]),$$

where $f^{-i}([0,1]) := \{x \in \mathbb{R} : f^i(x) \in [0,1]\}$ for each $i$. Moreover, $f(\Lambda) \subseteq \Lambda$, so $f|_{\Lambda}$ acts as a self-map on $\Lambda$. Thus, to study the states that have bounded orbits under $f$, it is enough to focus instead on the dynamical system $(\Lambda, f|_{\Lambda})$. At the cost of abusing our notation a bit, let us denote $f|_{\Lambda}$ simply by $f$, so this system is denoted as $(\Lambda, f)$. But how can one study a dynamical system with such a strange state space? Enter, symbolic dynamics.

Put $I_0 := [0,\alpha]$ and $I_1 := [1 - \alpha,1]$, and define the map $\varphi : \Lambda \to \{0,1\}^{\infty}$ by $\varphi(x) := (\omega_0, \omega_1, \ldots)$ where

$$\omega_i := \begin{cases} 0, & \text{if } f^i(x) \in I_0 \\ 1, & \text{if } f^i(x) \in I_1, \end{cases}$$

for each $i \in \mathbb{Z}_+$. This map is well-defined, because for any $x \in \Lambda$ and $m \in \mathbb{N}$, we have $f^m(x) \in [0,1]$, and this holds iff $f^{m-1}(x) \in I_0 \sqcup I_1$.

Let us check that $\varphi$ is a bijection. Take any $(\omega_0, \omega_1, \ldots) \in \{0,1\}^{\infty}$, and put $I(\omega_0, \ldots, \omega_m) := \bigcap_{i=0}^{m} f^{-i}(I_{\omega_i})$ for each $m \in \mathbb{Z}_+$. Clearly, $I(\omega_0) \supseteq I(\omega_0, \omega_1) \supseteq \cdots$ while each of these sets is closed (because $f$, and hence $f^2, f^3, \ldots$, are all continuous maps). We now come to the reason why we restrict $\lambda$ to exceed $2 + \sqrt{5}$. The derivative of $f$ on $I_0$ takes its smallest value at $\alpha$, and we have $f'(\alpha) = 2\lambda\sqrt{\frac{1}{4} - \frac{1}{\lambda}}$. This number is 1 precisely when $\lambda = 2 + \sqrt{5}$. So, the hypothesis $\lambda > 2 + \sqrt{5}$ ensures that there is a real number $\theta$ with $f'(x) \geq \theta > 1$ for every $x \in I_0$. In fact, by symmetry, we have $|f'(x)| \geq \theta > 1$ for every $x \in I_0 \sqcup I_1$. But then, for any $x, y \in I(\omega_0)$,
the Mean Value Theorem yields

$$|x - y| \leq \frac{1}{\theta} |f(x) - f(y)| \leq \frac{1}{\theta}$$

because \(x, y \in I(\omega_0)\) means that \(x\) and \(y\) lie in the same one of the intervals \(I_0\) and \(I_1\). Similarly, for any \(x, y \in I(\omega_0, \omega_1)\), the Mean Value Theorem yields

$$|x - y| \leq \frac{1}{\theta} |f(x) - f(y)| \leq \frac{1}{\theta^2} |f^2(x) - f^2(y)| \leq \frac{1}{\theta^2}$$

because \(x, y \in I(\omega_0, \omega_1)\) means that not only \(x\) and \(y\) lie in the same one of the intervals \(I_0\) and \(I_1\), but this is true also for \(f(x)\) and \(f(y)\). Proceeding this way inductively, therefore, we find that \(|x - y| \leq 1/\theta^m\) for every \(x, y \in I(\omega_0, ..., \omega_{m-1})\) and \(m \in \mathbb{N}\), and hence,

$$\text{diam}(I(\omega_0, ..., \omega_m)) \downarrow 0.$$  

We may then apply the Cantor-Fréchet Intersection Theorem to conclude that \(\bigcap_{m=0}^{\infty} I(\omega_0, ..., \omega_m)\) contains a unique point, say, \(x\). Given the arbitrary choice of \((\omega_0, \omega_1, ...)\) in \(\{0, 1\}^\infty\), the existence part of this finding proves the surjectivity of \(\varphi\), and the uniqueness part of it proves the injectivity of \(\varphi\).

It is easy to show that \(\varphi\) is continuous. After all, by definition of the product topology, any closed set in \(\{0, 1\}^\infty\) is the union of finitely many sets of the form \(\{\omega_0\} \times \cdots \times \{\omega_m\} \times \{0, 1\} \times \{0, 1\} \times \cdots\), where \(m \in \mathbb{Z}_+\) and \(\omega_0, ..., \omega_m \in \{0, 1\}\). But the inverse image of any such set under \(\varphi\) is closed in \(\Lambda\), because

$$\varphi^{-1}(\{\omega_0\} \times \cdots \times \{\omega_m\} \times \{0, 1\} \times \{0, 1\} \times \cdots) = I(\omega_0, ..., \omega_m).$$

Conclusion: \(\varphi\) is continuous. Since \(\Lambda\) is a closed, hence compact, subset of \([0, 1]\), the Homeomorphism Theorem in turn yields that \(\varphi\) is a homeomorphism from \(\Lambda\) onto \(\{0, 1\}^\infty\).

Finally, notice that for any \(x\) in \(\Lambda\), \(\varphi(x)\) is the binary sequence \((\omega_0, \omega_1, ...)\) where \(x \in I_{\omega_0}\), \(f(x) \in I_{\omega_1}\), \(f^2(x) \in I_{\omega_2}\) and so on, and hence

$$\varphi(f(x)) = (\omega_1, \omega_2, ...) = \sigma(\omega_0, \omega_1, ...) = \sigma(\varphi(x)),$$

where \(\sigma\) is the shift operator on \(\{0, 1\}^\infty\). Thus: \(\varphi\) is a topological conjugacy between \((\Lambda, f)\) and \((\{0, 1\}^\infty, \sigma)\), and hence, \((\Lambda, f) \cong (\{0, 1\}^\infty, \sigma)\).

This finding allows us to study the dynamics of the system \((\Lambda, f)\) by means of the system \((\{0, 1\}^\infty, \sigma)\). For instance, for any positive integer \(k\), we know from Example 1.6 that there are exactly \(2^k\) many points in \(\{0, 1\}^\infty\).
whose periods under $\sigma$ equal $k$. Given the analysis above, we can now conclude that for any $k \in \mathbb{N}$, there are exactly $k$ many real numbers $x$ such that $|\text{orb}_f(x)| = k$. (It would have been quite difficult to arrive at this conclusion by analyzing $(\mathbb{R}, f)$ directly.)

**Warning.** We have established the topological conjugacy of $(A, f)$ and $(\{0, 1\}^\infty, \sigma)$ under the assumption that $\lambda > 2 + \sqrt{5}$. In fact, this result remains valid for any $\lambda > 4$, but this more general case is much harder to prove. See, for instance, de Melo and van Strien (1993).

**Exercises**

1.1. Let $a, b, c$ and $\alpha$ be real numbers, and consider the dynamical systems $(\mathbb{R}, f_{a, b, c})$ and $(\mathbb{R}, g_{a})$ where $f_{a, b, c}(x) := ax^2 + bx + c$ and $g_{a}(x) := x^2 + \alpha$. Show that if $\alpha = ac + \frac{b}{2} - \frac{b^2}{4a}$, there is an affine self-map on $\mathbb{R}$ which is a topological conjugacy between $(\mathbb{R}, f_{a, b, c})$ and $(\mathbb{R}, g_{a})$. (Thus, the dynamics entailed by any 3-parameter system of the form $(\mathbb{R}, f_{a, b, c})$ can be understood completely by studying those of a 1-parameter system $(\mathbb{R}, g_{a})$.)

1.2. Verify that $(\{0, 1\}^\infty, f) \cong (C, g)$, as asserted in Example 1.10.

1.3. Consider the topological Markov chain $(X_A, \sigma|_{X_A})$ we have introduced in Example 1.7.

a. Show that $A^m \in \mathbb{R}_+^{n \times n}$ implies $A^k \in \mathbb{R}_+^{n \times n}$ for any two positive integers $m$ and $k$ with $k \geq m$.

b. Assume that $A$ is irreducible, and use part (a) to show that the set of all points in $X_A$ that are periodic under $\sigma|_{X_A}$ is dense in $X_A$.

c. Give an example to show that the previous conclusion would be false without the irreducibility hypothesis.

1.4. Consider the dynamical system $([0, 1], f)$ where $f(x) := 1 - |2x - 1|$. (This map is known as the **tent map** in the theory of dynamical systems; see Figure 8.6.)

a. For any $x \in [0, 1]$, prove that $x$ is eventually periodic under $f$ iff $x$ is a rational number.

b. For any $x \in [0, 1]$, prove that $x$ is periodic under $f$ iff $x$ can be expressed as $\frac{m}{n}$ where $m$ is an even integer and $n$ an odd integer.

c. Show that the self-map $x \mapsto (\sin \frac{\pi x}{2})^2$ on $[0, 1]$ is a topological conjugacy between $([0, 1], f)$ and $([0, 1], g)$, where $g(x) := 4x(1 - x)$.

1.5. Let $(\mathbb{R}, f)$ be a dynamical system such that $f$ is a homeomorphism. Show that there is no real number $x$ whose period under $f$ is at least 3.

1.6. Let $(X, f)$ be a dynamical system such that $f$ is injective. Prove that a point $x$ in $X$ is periodic under $f$ iff it is eventually periodic under $f$. 

13
1.7. Let \((X, f)\) and \((Y, g)\) be two dynamical systems. A map \(\varphi : X \to Y\) is said to be a **topological semiconjugacy** between \((X, f)\) and \((Y, g)\) if \(\varphi\) is a continuous surjection and \(\varphi \circ f = g \circ \varphi\). If there exists such a map we say that \((Y, g)\) is a **factor** of \((X, f)\).

**a.** Let \(\varphi\) be a topological semiconjugacy between \((X, f)\) and \((Y, g)\), and define the self-map \(h\) on \(X/\varphi\) by \(h([x]_{\varphi}) = [f(x)]_{\varphi}\). Show that \((X/\varphi, h) \cong (Y, g)\).

**b.** For any \(x \in [0, 1)\), put \(d_1(x) := 0\) if \(x \in [0, \frac{1}{2})\), and \(d_1(x) := 1\) otherwise. If \(d_1(x) = 0\), then put \(d_2(x) := 0\) if \(x \in [0, \frac{1}{2})\), and \(d_2(x) := 1\) otherwise, and if \(d_1(x) = 1\), then put \(d_2(x) := 0\) if \(x \in \left[\frac{1}{2}, \frac{3}{4}\right)\), and \(d_2(x) := 1\) otherwise. Continuing this way yields a sequence \((d_m(x))\) in \([0, 1]^\infty\), which is called the **dyadic expansion** of \(x\). Use induction to show that

\[
\sum_{i=1}^{m} d_i(x)2^{-i} \leq x < \sum_{i=1}^{m} d_i(x)2^{-i} + 2^{-m},
\]

and hence \(x = \sum_{i=1}^{\infty} d_i(x)2^{-i}\).

**c.** (**Symbolic Dynamics of the Doubling Map**) Show that \(\varphi : \{0, 1\}^\infty \to X\), defined by \(\varphi(a_1, a_2, \ldots) := \sum_{i=1}^{\infty} a_i2^{-i}\), is a topological semiconjugacy between \((\{0, 1\}^\infty, \sigma)\) and \((X, f)\), where \((X, f)\) is the dynamical system introduced in Example 1.5.

## 2 Topological Transitivity and Minimality

### 2.1 Topological Transitivity

**Topologically Transitive Dynamical Systems**

Suppose we are given a dynamical system \((X, f)\) that describes the transitions of a particle from one physical state to another, but we are able to identify the initial state of the system only up to a certain error. It then makes sense to specify a (presumably small) open neighborhood, say, \(O\), of where we think the initial state is, and then to investigate how \(f\) “moves” that entire neighborhood about the state space. One immediate question that arises in this scenario is if our neighborhood \(O\) is largely stagnant or instead it travels a lot. Topologically speaking, we may formalize the idea that “\(O\) travels a lot” by requiring that for any nonempty open subset \(U\) of \(X\), however small, at least one point in \(O\) would eventually enter in \(U\), that is, \(f^k(x) \in U\) for some \((x, k) \in (O, \mathbb{N})\).

Now, of course, all this depends on what \(O\) actually is. But there are dynamical systems in which all nonempty open sets “travel a lot,” and for those systems, certain questions about the limit behavior of the solution...
trajectory may be accurately answered even if we only have a very rough idea about the initial state of the system. Such systems are special, and are given a special name:

**Definition.** A dynamical system \((X, f)\) is said to be **topologically transitive** if for every nonempty open subsets \(O\) and \(U\) of \(X\), there exists a nonnegative integer \(k\) such that \(f^k(O) \cap U \neq \emptyset\). (See Figure 8.7.)

**Warning.** Suppose \((X, f)\) is topologically transitive, and take any nonempty \(O, U \in O_X\). Then, \(f^{-1}(U)\) is open in \(X\), so there exists a \(k \in \mathbb{Z}_+\) such that there is a state \(x\) in \(f^k(O) \cap U\). But then \(f(x) \in f^{k+1}(O) \cap U\), that is, \(f^{k+1}(O) \cap U \neq \emptyset\). Thus, we can take \(k\) to be a *positive* integer in the definition above.

**Remark.** Our definition formulates the notion of topological transitivity in terms of the forward travel of a given neighborhood \(O; U\); but (given the arbitrariness of \(O\) and \(U\)) we can give an equivalent formulation in terms of backward travels as well. To wit, a dynamical system \((X, f)\) is topologically transitive iff for every nonempty open subsets \(O\) and \(U\) of \(X\), there exists a nonnegative integer \(k\) such that \(f^{-k}(O) \cap U \neq \emptyset\). (Proof. To see the “if” part of the claim, take any nonempty \(O, U \in O_X\), and note that the topological transitivity of \((X, f)\) entails \(f^k(U) \cap O \neq \emptyset\) for some \(x \in U\). Then, there is a \(y \in O\) such that \(y = f^k(x)\) for some \(x \in U\). This implies that \(x \in f^{-k}(O)\), that is, \(f^{-k}(O) \cap U \neq \emptyset\). The “only if” part of our claim is analogously proved.)

Topological transitivity is an intrinsic characteristic of dynamical systems in that it is preserved by topological conjugacy. (That is, if \((X, f) \cong (Y, g)\) and \((X, f)\) is topologically transitive, then \((Y, g)\) is topologically transitive as well.) As we shall see, this often helps one in establishing whether a given system is topologically transitive or not.

**Transitive States of a Dynamical System**

Let \((X, f)\) be a dynamical system. It is natural to think of this system moving a state \(x\) “a lot” if the trajectory of \(x\) under \(f\) passes through every nonempty open subset of \(X\), which is the same thing as saying that \(\text{orb}_f(x)\) is dense in \(X\). Such a point \(x\) in \(X\) is said to be **\(f\)-transitive**. (See Figure 8.8 and compare with Figure 8.7.)

In general, as the following example demonstrates, the existence of an \(f\)-transitive state in a dynamical system is not enough to ensure the topological transitivity of that system.
Example 2.1. Let $X$ stand for the subspace $\{0, 1, \frac{1}{2}, \frac{1}{3}, \ldots\}$ of $\mathbb{R}$, and consider the self-map $f$ on $X$ defined by $f(0) := 0$ and $f \left( \frac{1}{m} \right) := \frac{1}{m+1}$ for each positive integer $m$. Then, $(X, f)$ is a dynamical system, and 1 is an $f$-transitive point of $X$, because $\text{orb}_{f}(1) = \{1, \frac{1}{2}, \frac{1}{3}, \ldots\}$. (Note. 1 is the only $f$-transitive point of $X$.) And yet $(X, f)$ is not topologically transitive. For instance, $\{\frac{1}{2}\}$ and $\{1\}$ are open subsets of $X$ with $f^{k}(\{\frac{1}{2}\}) \cap \{1\} = \emptyset$ for every $k \geq 1$.

The difficulty encountered in this example is due to the presence of isolated points in the state space $X$ of the system. If there are no such points in $X$, then a nonempty finite subset of $X$ cannot be open. (Proof. Suppose $S$ is a finite open subset of $X$ with $|S| \geq 2$, and $x$ any point in $S$. Since $X$ is a $T_{1}$-space, for every $y \in S \setminus \{x\}$, there is an $O_{y} \in \mathcal{O}_{X}(x)$ that does not contain $y$. But then $O := \bigcap \{O_{y} : y \in S \setminus \{x\}\} \in \mathcal{O}_{X}$, so since $\{x\} = S \cap O$, we must conclude that $\{x\}$ is open in $X$, that is, $x$ is an isolated point of $X$.) This fact yields a very nice sufficient condition for topological transitivity.

Lemma 2.1. (Silverman) Let $(X, f)$ be a dynamical system such that there are no isolated points in $X$. If there is an $f$-transitive point in $X$, then $(X, f)$ is topologically transitive.

Proof. Assume that $\text{orb}_{f}(x)$ is dense in $X$ for some $x \in X$, and take any $O, U \in \mathcal{O}_{X}$. It is plain that $y := f^{l}(x) \in O$ for some positive integer $l$. But, since no singleton is open in $X$ by hypothesis, and $X$ is a $T_{1}$-space, no nonempty finite set in $X$ is open. (Why?) In particular, $U$ is an infinite set, and hence $U \setminus \{x, f(x), \ldots, f^{l}(x)\}$ is a nonempty open subset of $X$. As $\text{orb}_{f}(x)$ is dense in $X$, then, there must be a $k \in \{l+1, l+2, \ldots\}$ such that $f^{k}(x) \in U$. But then $f^{k-1}(y) \in U$, that is, $f^{k-1}(O) \cap U \neq \emptyset$.

There are alternative scenarios under which the existence of a dense orbit implies topological transitivity. In particular, this happens whenever the transition function under consideration is surjective.

Lemma 2.2. Let $(X, f)$ be a dynamical system such that $f$ is a surjection. Suppose that there is an $f$-transitive point in $X$. Then,

(a) every $f$-invariant open subset of $X$ is dense in $X$; and
(b) $(X, f)$ is topologically transitive.

Proof. (a) Assume that $\text{orb}_{f}(x)$ is dense in $X$ for some $x \in X$, and suppose $V$ is a nonempty open set in $X$ with $f(V) \subseteq V$. Then, there is a nonnegative integer $k$ such that $f^{k}(x) \in V$. Since $f(V) \subseteq V$, we find
$f^m(x) \in V$ for every integer $m \geq k$ (by induction). If $k = 0$, we are done (because this means that $\text{orb}_f(x)$ is contained in $V$). Suppose $k > 0$, and note that $\{x, ..., f^{k-1}(x)\} \cup V$ contains $\text{orb}_f(x)$. Taking the closure of these sets, and recalling that $\text{orb}_f(x)$ is dense in $X$ (and that finite sets in $X$ are closed), therefore, we find

$$\{x, ..., f^{k-1}(x)\} \cup \text{cl}(V) = X.$$  

Then, applying $f$ to both sides of this equation, and recalling that $f(\text{cl}(V)) \subseteq \text{cl}(f(V)) \subseteq \text{cl}(V)$ and $f(X) = X$, we find

$$\{f(x), ..., f^k(x)\} \cup \text{cl}(V) = X.$$  

Proceeding by induction, therefore,

$$\{f^k(x), ..., f^{2k-1}(x)\} \cup \text{cl}(V) = X.$$  

Since $f^k(x), ..., f^{2k-1}(x) \in V$, this means that $\text{cl}(V) = X$.

(b) Take any nonempty open sets $O$ and $U$ in $X$. Put $V := O \cup f(O) \cup f^2(O) \cup \cdots$, and note that $f(V) \subseteq V$. Thus, $V$ is dense in $X$ by part (a), and hence it intersects $U$. But this means that $f^k(O) \cap U \neq \emptyset$ for some $k \in \mathbb{Z}_+$.

Let us now look into the reverse direction. First, the bad news: Topological transitivity of a dynamical system does not guarantee the existence of a transitive state. (We will demonstrate this below in a little while.) But in most instances of interest things go well. To see what we mean by this, take any dynamical system $(X, f)$ and a state in $x \in X$. It follows easily from the definitions that $x$ is $f$-transitive iff it belongs to $\bigcap \{\bigcup_{k=0}^{\infty} f^{-k}(O) : O \in \mathcal{O}_X\}$. (Verify!) Then, if $\mathcal{B}_X$ is a basis for the topology of $X$, we have

$$\text{the set of all } f\text{-transitive points in } X = \bigcap_{B \in \mathcal{B}_X} \bigcup_{k=0}^{\infty} f^{-k}(B).$$  

This way of looking at things allows us make a Baire category type argument as follows.

**Lemma 2.3.** (Silverman) Let $(X, f)$ be a dynamical system such that $X$ is a second-countable Baire space. If $(X, f)$ is topologically transitive, then the set of all $f$-transitive points in $X$ is a dense $G_\delta$-subset of $X$.

**Proof.** Assume that $(X, f)$ is topologically transitive. Take a countable basis for the topology of $X$, and enumerate it as $\{O_1, O_2, \ldots\}$. Then,

$$\text{the set of all } f\text{-transitive points in } X = \bigcap_{i=1}^{\infty} \bigcup_{k=0}^{\infty} f^{-k}(O_i).$$  

17
But for each $i$, the set $\bigcup_{k=0}^{\infty} f^{-k}(O_i)$ is open (because $f$ is continuous), and dense (because $(X, f)$ is topologically transitive). Since $X$ is Baire, it follows that $\bigcap_{i=1}^{\infty} \bigcup_{k=0}^{\infty} f^{-k}(O_i)$ is a dense subset of $X$.

**Warning.** In view of the Baire Category Theorem, we can take $X$ as a complete and separable metric space in Lemma 2.3. Alternatively, thanks to Theorem 4.4 of Chapter 7, we can assume that it as a second-countable and locally compact Hausdorff space.

The following combines these lemmata into a characterization theorem.

**Theorem 2.4.** Let $(X, f)$ be a dynamical system such that $X$ is a second-countable Baire space. In addition, suppose that either $X$ has no isolated points or $f$ is surjective. Then, $(X, f)$ is topologically transitive if, and only if, there is an $f$-transitive point in $X$.

In particular, if $X$ is a compact and connected metric space, which is a fairly common setting in which topological dynamics are studied in practice, then $(X, f)$ is topologically transitive iff there is an $f$-transitive point in $X$. (The same conclusion is holds if instead of connectedness we assume that $f$ is surjective.)

**Warning.** Presumably due to the previous observations, many authors define a dynamical system to be “topologically transitive” when there is at least one state in the system with a dense orbit. The terminology that we use here is, however, more commonly adopted.

**Examples**

**Example 2.2.** $(X, \text{id}_X)$ is a dynamical system which is not topologically transitive for any Hausdorff space $X$ with $|X| \geq 2$.

**Example 2.3.** Let $(X, f)$ be the dynamical system considered in Example 1.2. This system is topologically transitive iff $f$ is an irreducible permutation on $X$. Similarly, there is an $f$-transitive state in $X$ iff $f$ is an irreducible permutation on $X$.

**Example 2.4.** Let $X$ be a complete and separable metric space with no isolated points, and assume that $|X| > 1$. (For instance, $X$ can be any non-trivial separable Banach space.) Let $f$ be a contraction on $X$, and consider the dynamical system $(X, f)$. The fixed point $x^*$ of $X$ is obviously not $f$-transitive (because $X$ has at least two points in it). Now take any
Example 2.5. (Rotations and Translations, Again) Consider the dynamical system \((X, f_\alpha)\) introduced in Example 1.4. In view of Theorem 2.4, this system is topologically transitive iff there is an \(f\)-transitive point in \(X\). But we have seen in Example 1.4 that every point in \(X\) is periodic under \(f_\alpha\) when \(\alpha\) is rational, and \(\text{orb}_{f_\alpha}(x)\) is dense for every \(x \in X\) when \(\alpha\) is irrational. Thus: \((X, f_\alpha)\) is topologically transitive iff \(\alpha\) is irrational.

Since topological transitivity is preserved under topological conjugacy, this finding settles the matter for rotation systems on the circle as well (Example 1.9). That is, \((S^1, g_\alpha)\) is topologically transitive iff \(\alpha\) is an irrational number in \([0, 1)\).

Example 2.6. (The Doubling Map, Again) Let \((X, f)\) be the dynamical system introduced in Example 1.5 (where \(f\) is the doubling map). Note that the length of the image of any subinterval of \([0, \frac{1}{2})\), or of \([\frac{1}{2}, 1)\), under \(f\) is twice as large as that interval. Consequently, if \(I\) is a nondegenerate interval with \(0 \in I\), it is clear that \(f^k(I) = [0, 1)\) for some \(k \in \mathbb{N}\). Now, take any nondegenerate open subinterval \(J\) of \([0, 1)\). If \(\frac{1}{2}\) is in \(J\), then \(f^2(J)\) is a set that contains a nondegenerate interval that contains \(0\), so \(f^{k+2}(J) = [0, 1)\) for some \(k \in \mathbb{N}\). If \(\frac{1}{2}\) is not in \(J\), then either \(J \subseteq [0, \frac{1}{2})\) or \(J \subseteq [\frac{1}{2}, 1)\), so there is an \(l \in \mathbb{N}\) such that \(f(J)\) is twice as long as \(J\). It follows that there is positive integer \(l\) such that \(f^l(J)\) is a set that contains (a nondegenerate interval that contains) \(\frac{1}{2}\), so as before, \(f^{k+l+2}(J) = [0, 1)\). Conclusion: For every nondegenerate subinterval \(J\) of \([0, 1)\), we have \(f^m(J) = [0, 1)\) for some positive integer \(m\), and this implies that \((X, f)\) is topologically transitive.

Example 2.7. We are now ready to show that topological transitivity of a dynamical system does not guarantee the existence of a transitive state. Let \((X, f)\) be the dynamical system studied in the previous example (where \(f\) is the doubling map). Put \(Y := [0, 1) \cap \mathbb{Q}\), and view this set as a metric
subspace of $X$. Obviously, $Y$ is $f$-invariant, so $(Y, g)$ is a dynamical system where $g := f|_Y$. We have seen in Example 1.5 that $Y$ consists of all states in $X$ that are eventually periodic under $f$. Thus, the orbit of every state in $Y$ under $g$ is finite, and hence, not dense in $Y$. Conclusion: No state in $Y$ is $g$-transitive.

We now show that $(Y, g)$ is topologically transitive. (Thus, the requirement of being Baire cannot be omitted in Lemma 2.3.) Take any open sets $O$ and $U$ in $Y$. Since $Y$ is a subspace of $X$, $O = O' \cap Y$ and $U = U' \cap Y$ for some $O', U' \in \mathcal{O}_X$. Since we have seen above that $(X, f)$ is topologically transitive, there is a $k \in \mathbb{Z}_+$ with $f^k(O') \cap U' \neq \emptyset$. Moreover, since $Y$ is dense in $X$ and $f$ is an open map (relative to the Kronecker metric), there is a state $y$ in $Y$ that belongs to this intersection, that is, $y \in f^k(O')$ and $y \in U'$. Since $y \in Y$, the latter statement entails that $y \in U$. Moreover, $y = f^k(x)$ for some $x \in O'$. But then, since $y$ has a finite orbit under $f$, so must $x$, that is, $x \in Y$. It follows that $x \in O$, and hence, $y \in f^k(O)$. Thus, $f^k(O) \cap U$, which equals $g^k(O) \cap U$, is nonempty. In view of the arbitrary choice of the sets $O$ and $U$, therefore, we conclude that $(Y, g)$ is topologically transitive.

**Example 2.8. (Shifts, Again)** Let $\Omega$ be a finite set, say, $\Omega := \{\omega_1, \ldots, \omega_n\}$, and consider the dynamical system $(\Omega^\infty, \sigma)$, where $\sigma$ is the shift operator on $\Omega^\infty$. Let us call a string of $k$ many elements of $\Omega$ (allowing repetitions) a $k$-block. For instance, $\omega_i$ is a 1-block for any $i = 1, \ldots, n$, and $\omega_i, \omega_j$ is a 2-block for any $i, j = 1, \ldots, n$, and so on. Now let $x$ be a sequence in $\Omega$ such that the first $n$ terms of $x$ are made up of the distinct 1-blocks, the next $2n^2$ terms of $x$ are made up of the distinct 2-blocks, and the next $3n^3$ terms of $x$ are made up of the distinct 3-blocks, and so on. (For instance, when $n = 2$, the sequence $x$ may look like

$$(\omega_1, \omega_2, \omega_1, \omega_1, \omega_1, \omega_2, \omega_2, \omega_1, \omega_2, \omega_2, \omega_1, \omega_1, \omega_1, \ldots, \omega_2, \omega_2, \omega_2, \omega_2, \ldots).$$

Then, the orbit of $x$ under $\sigma$ is dense in $\Omega^\infty$. Indeed, a basis for the topology of $\Omega^\infty$ consists of all sets of the form $\{\omega_1\} \times \cdots \times \{\omega_k\} \times \Omega \times \Omega \times \cdots$, where $k \in \mathbb{N}$ and $(\omega_1, \ldots, \omega_k) \in \Omega^k$. But, for any such $k$ and $(\omega_1, \ldots, \omega_k)$, it is clear that applying $\sigma$ to $x$ sufficiently many times would yield a sequence whose first $k$ terms would match the vector $(\omega_1, \ldots, \omega_k)$ exactly. It follows that $\text{orb}_\sigma(x)$ intersects every element of a basis for the product topology on $\Omega^\infty$, that is, $\text{orb}_\sigma(x)$ is dense in $\Omega^\infty$. Since $\sigma$ is surjective, then, we may apply Lemma 2.2 to conclude that $(\Omega^\infty, \sigma)$ is topologically transitive.

**Example 2.9. (Topological Markov Chains, Again)** Take any positive integer $n \geq 2$, and an $n \times n$ matrix $A \in \{0, 1\}^{n \times n}$ such that at least one entry of
is topologically transitive, provided that

introduced in Example 1.7. We will show here that this dynamical system

column. Then, consider the topological Markov chain

\(A\) is 1, and that if the \(j\)th row of \(A\) consists of all 0s, then so does its \(j\)th
column. Then, consider the topological Markov chain

A

is 1, and that if the

is an

X

difficult to check that this implies \(a_{ij}^{(m)} > 0\) for each \(m = M, M + 1, \ldots\)

and

i, j = 1, ..., n.\) The following claim lies at the heart of our argument.

Claim. For any positive integer \(M\), we have \(a_{ij}^{(M)} > 0\) iff there is a \(z\) in

\(X_A\) with \(z_1 = \omega_i\) and \(z_{M+1} = \omega_j\).

Proof. For \(M = 1\), the claim is immediate from the definition of \(X_A\).

As the induction hypothesis, suppose that it is true for an arbitrarily fixed

\(M \in \mathbb{N}\). To prove that our claim is valid for \(M + 1\), note that

\[ a_{ij}^{(M+1)} = \sum_{k=1}^n a_{ik}^{(M)} a_{kj}, \]

so \(a_{ij}^{(M+1)} > 0\) iff there is a \(k \in \{1, \ldots, n\}\) such that \(a_{ik}^{(M)} > 0\)

and \(a_{kj} = 1\). By our induction hypothesis, therefore, \(a_{ij}^{(M+1)} > 0\) iff there

exist \(k \in \{1, \ldots, n\}\) and \(x, y \in X_A\) with \(x_1 = \omega_1\) and \(x_{M+1} = \omega_k\), and \(y_1 = \omega_k\)

and \(y_2 = \omega_j\). But, clearly, the latter statement holds iff

\[(x_1, \ldots, x_{M+1}, y_2, y_3, \ldots) \in X_A,\]

that is, there is a \(z \in X_A\) with \(z_1 = \omega_i\) and \(z_{M+2} = \omega_j\).

Assume now that \(A\) is irreducible, and take any open sets \(O\) and \(U\) in

\(X\) that intersect \(X_A\). We may choose \(O\) and \(U\) as

\[ O = \{\omega_{i_1}\} \times \cdots \times \{\omega_{i_m}\} \times \{0, 1\}^\infty \times \cdots \]

and

\[ U = \{\omega_{j_1}\} \times \cdots \times \{\omega_{j_n}\} \times \{0, 1\}^\infty \times \cdots \]

for some \(m, n \in \mathbb{N}\), \(\omega_{i_1}, \ldots, \omega_{i_m}, \omega_{j_1}, \ldots, \omega_{j_n} \in \Omega\), as such sets form a basis for the topology of \(X\). Now, since \(A\) is irreducible, there is a positive integer \(M\)

such that \(a_{i_m j_1}^{(M)} > 0\), so by the Claim above, there is a \(z \in X_A\) with \(z_1 = \omega_{i_m}\)

and \(z_{M+1} = \omega_{j_1}\). Then,

\[ w := (\omega_{i_1}, \ldots, \omega_{i_m}, z_2, \ldots, z_M, \omega_{j_1}, \ldots, \omega_{j_n}, z_{n+1}, z_{n+2}, \ldots) \in X_A.\]

Since \(w \in O \cap X_A\) while \(\sigma^{m+M-1}(w) \in U \cap X_A\), we conclude that \(\sigma^{m+M-1}(O)\)

intersects \(U\). In view of the arbitrary choice of \(O\) and \(U\), therefore, we con-
clude that \((X_A, \sigma|_{X_A})\) is topologically transitive.
Warning. Under the assumption that for every \( i \in \{1, \ldots, n\} \), there exists an \( x \in X_A \) with \( x_k = \omega_i \) for some \( k \in \mathbb{N} \), irreducibility of \( A \) is also necessary for \((X_A, \sigma|_{X_A})\) being topologically transitive. Indeed, suppose \( A \) is not irreducible. Then, there exist \( i, j \in \{1, \ldots, n\} \) such that \( a_{ij}^{(m)} = 0 \) for every \( m \in \mathbb{N} \). Put \( O := \{\omega_i\} \times \{0,1\}^\infty \times \cdots \) and \( U := \{\omega_j\} \times \{0,1\}^\infty \times \cdots \). Since \( \sigma(X_A) \subseteq X_A \), the assumption we just made entails that there exist members of \( X_A \) whose first terms are \( \omega_i \) and \( \omega_j \), respectively. Thus, \( O \cap X_A \) and \( U \cap X_A \) are nonempty sets in \( X_A \). But for any positive integer \( m \), the Claim above entails that there is no \( z \) in \( X_A \) such that \( z_1 = \omega_i \) and \( z_{m+1} = \omega_j \), so \( \sigma^m(O \cap X_A) \) does not intersect \( U \cap X_A \). Thus, \((X_A, \sigma|_{X_A})\) is not topologically transitive.

**Topologically Transitivity as an “Irreducibility Property”**

What we have found in the previous example is quite telling. Indeed, it makes sense to think of topological transitivity as a general “irreducibility” property for dynamical systems at large. For instance, it is plain that if \((X, f)\) is topologically transitive, then there cannot be two disjoint open \( f \)-invariant sets in \( X \). In other words, in that case we cannot find two separate subsystems of \((X, f)\), say, \((O, f|_O)\) and \((U, f|_U)\) with \( O \) and \( U \) being disjoint members of \( \mathcal{O}_X \).

Less trivial is the following observation which again speaks to topological transitivity being an “irreducibility” property.

**Proposition 2.5.** Let \((X, f)\) be a dynamical system such that \( X \) is a second-countable Baire space, and \( Y \) a closed \( f \)-invariant subset of \( X \). If \((X, f)\) is topologically transitive, then either \( X \setminus Y \) is a finite set or \( \text{int}(Y) = \emptyset \). If, in addition, \( X \) has no isolated points, then either \( Y = X \) or \( \text{int}(Y) = \emptyset \).

**Proof.** Suppose \((X, f)\) is topologically transitive, and \( U := \text{int}(Y) \neq \emptyset \). Then, by Lemma 2.3, there is an \( x \in X \) such that \( \text{orb}_f(x) \) is dense in \( X \). So, \( \text{orb}_f(x) \) intersects \( U \), that is, \( f^k(x) \in U \) for some \( k \in \mathbb{Z}_+ \). Since \( Y \) is \( f \)-invariant, we also have \( \{f^k(x), f^{k+1}(x), \ldots\} \subseteq Y \), that is, \( Y \) contains all but finitely many elements of \( \text{orb}_f(x) \). Adding these elements to \( Y \), therefore, we obtain a closed and dense subset of \( X \), which is must then equal \( X \). The second assertion also follows from this argument, for if there are no isolated points in \( X \), taking finitely many points out of a dense subset of \( X \), namely, \( \text{orb}_f(x) \), yields again a dense set in \( X \), which then means that \( Y \) contains a dense subset of \( X \). As \( Y \) is closed in \( X \), it follows that \( Y = X \).

### 2.2 Sensitive Dependence on Initial Conditions

One of the main themes of the theory of dynamical systems is the possibility – in fact, commonality – of the phenomenon that a given system might lead
two initial states that are arbitrarily close to each other to drastically different future states. Thus, even though the underlying model is deterministic, in most cases of interest we have to live with the fact that “the future is hardly predictable,” because it is likely that the initial state of the system would be known only up to a small (estimation and/or measurement) error.\(^2\)

There are different ways of formulating what it means for a dynamical system to exhibit such sensitivity to the description of its initial state. A particularly appealing one can be formulated in the context of metric spaces as follows:

**Definition.** Let \( (X, f) \) be a dynamical system where \( X \) is a metric space (with metric \( d \)). We say that this system exhibits **sensitive dependence on initial conditions** if there is a real number \( \delta > 0 \) such that for every \( x \in X \) and \( O \in \mathcal{O}_X(x) \), there is a state \( y \) in \( O \) and a nonnegative integer \( m \) such that \( d(f^m(x), f^m(y)) \geq \delta \).

**Warning.** Sensitive dependence on initial conditions is an inherently metric property, and as such, it is not preserved under topological conjugacy. For instance, \( ((1, \infty), f) \cong (\mathbb{R}_{++}, g) \), where \( f(x) := 2x \) and \( g(x) := x + 2 \), but while the former system exhibits sensitive dependence on initial conditions, the latter does not. As you are asked to prove in Exercise 2.13, however, this difficulty disappears when the state spaces of the dynamical systems at hand are compact metric spaces.

How common is sensitive dependence on initial conditions? The answer is surprising. It turns out that if a dynamical system (with an infinite state space) is “irreducible” in the sense of satisfying topological transitivity, and contains a dense collection of “well-behaved” states in the sense of having periodic orbits, then it must be “ill-behaved” in the sense of being sensitive to initial conditions. Put formally:

**Theorem 2.6.** Let \( (X, f) \) be a dynamical system such that \( X \) is a metric space. If \( (X, f) \) is topologically transitive and \( \text{Per}(X, f) \) is dense in \( X \), then either \( X \) is finite or \( f \) exhibits sensitive dependence on initial conditions.

In his popular 1989 introduction to the theory of dynamical systems, Robert Devaney has defined a dynamical system \((X, f)\), with \( X \) being a

\(^2\)In the popular science literature, this phenomenon is often referred to as the **butterfly effect**, a phrase coined by the mathematician/meteorologist Edward Lorenz. It corresponds to the (metaphorical) possibility that the structure of a tornado may be altered significantly by a minor modification of the initial state of the system that arise due to, say, the flapping of the wings of a butterfly on the other side of the globe.
metric space, to be chaotic as one that exhibits, “irreducibility,” “well-behavedness,” and “ill-behavedness” insofar these notions are captured by topological transitivity, existence of a dense set of periodic points, and sensitive dependence on initial conditions, respectively. But then Banks, et al. (1992) have proved the result we have stated above as Theorem 2.6, thereby establishing an unexpected redundancy in Devaney’s definition. Today, one refers to $(X, f)$, with $X$ being a metric space, as chaotic in the sense of Devaney if $(X, f)$ is topologically transitive and Per$(X, f)$ is dense in $X$.

Warning. There are other definitions of “chaotic behavior” that are in use. For instance, a dynamical system $(X, f)$, with $X$ being a metric space, is said to be chaotic in the sense of Auslander-Yorke if it is topologically transitive and $f$ exhibits sensitive dependence on initial conditions. This is a (strictly) weaker property than being chaotic in the sense of Devaney.

Proof of Theorem 2.6. Assume that $X$ is an infinite set. Since Per$(X, f)$ is dense in $X$, therefore, it is an infinite set as well. It follows that there are two members of Per$(X, f)$, say, $z$ and $w$, with disjoint orbits under $f$. (For, the orbit of every member of Per$(X, f)$ is finite, and if the orbits of two members of Per$(X, f)$ intersect, then these orbits must be the same.) Put

$$\gamma = \min\{d(\omega, \nu) : (\omega, \nu) \in \text{orb}_f(z) \times \text{orb}_f(w)\},$$

and note that $\gamma$ is a (well-defined) positive number (because both $\text{orb}_f(z)$ and $\text{orb}_f(w)$ are finite sets). Then, any $x$ in $X$ is at least $\gamma/2$ away from either $\text{orb}_f(z)$ or $\text{orb}_f(w)$. (For, otherwise, $d(x, \omega) < \gamma/2$ and $d(x, \nu) < \gamma/2$, and hence $d(\omega, \nu) < \gamma$, for some $(\omega, \nu) \in \text{orb}_f(z) \times \text{orb}_f(w)$, contradicting the definition of $\gamma$.) Thus:

Claim 1. There is a $\gamma > 0$ such that for every $x \in X$, there exists a $p_x \in \text{Per}(X, f)$ with

$$\text{dist}(x, \text{orb}_f(p_x)) \geq \frac{\gamma}{2}.$$

Now put $\delta := \gamma/8$, and take any $x \in X$ and $O \in \mathcal{O}_X(x)$. Clearly,

$$U := O \cap B(x, \delta)$$

is a nonempty open subset of $X$. Since Per$(X, f)$ is dense in $X$, there is a $p$ in $U$ which is periodic under $f$. On the other hand, by Claim 1, there is a $p_x \in \text{Per}(X, f)$ with $d(x, \omega) \geq 4\delta$ for every $\omega \in \text{orb}_f(p_x)$. Let $n$ stand for the period of $p$ under $f$, and put

$$V := \bigcap_{i=0}^{n-1} f^{-i}(B(f^{i}(p_x), \delta)).$$

Clearly, $V$ is an open neighborhood of $p_x$ in $X$. Moreover:

Claim 2. $f^{\ell}(V) \subseteq B(f^{\ell}(p_x), \delta)$ for any $\ell \in \{0, ..., n\}$.
Proof of Claim 2. Take any \( \ell \in \{0, ..., n\} \), and note that
\[
 f^\ell(V) = f^\ell \left( \bigcap_{i=0}^{n} f^{-i}(B(f^i(p_x), \delta)) \right) \subseteq \bigcap_{i=0}^{n} f^{-i+\ell}(B(f^i(p_x), \delta)).
\]
Since \( f^{-i+\ell}(B(f^i(p_x), \delta)) = B(f^\ell(p_x), \delta) \) when \( i = \ell \), therefore, we find \( f^\ell(V) \subseteq B(f^\ell(p_x), \delta) \), as we sought.

Now, since \((X, f)\) is topologically transitive, there is a nonnegative integer \( k \) such that \( f^k(U) \cap V \neq \emptyset \), that is, \( f^k(z) \in V \) for some \( z \in U \). We put \( j := \lfloor \frac{k}{n} \rfloor + 1 \), and note that \( \ell := nj - k \) is an integer in \( \{0, ..., n\} \). By Claim 2, therefore,
\[
 f^{nj}(z) = f^\ell(f^k(z)) \in f^\ell(V) \subseteq B(f^\ell(p_x), \delta).
\]
But since \( f^\ell(p_x) \in \text{orb}_f(p_x) \), applying the triangle inequality,
\[
 4\delta \leq d(x, f^\ell(p_x)) \leq d(x, p) + d(p, f^{nj}(z)) + d(f^{nj}(z), f^\ell(p_x)),
\]
and hence, given that \( f^{nj}(p) = p \), we find
\[
 d(f^{nj}(p), f^{nj}(z)) = d(p, f^{nj}(z)) > 4\delta - \delta - \delta = 2\delta.
\]
As \( d(f^{nj}(p), f^{nj}(z)) \leq d(f^{nj}(p), f^{nj}(x)) + d(f^{nj}(x), f^{nj}(z)) \), therefore,
\[
 \min\{d(f^{nj}(p), f^{nj}(x)), d(f^{nj}(x), f^{nj}(z))\} > \delta.
\]
Thus, for either \( y = z \) or \( y = p \), we have \( y \in O \) and \( d(f^{nj}(y), f^{nj}(x)) > \delta \).

2.3 Digression: Chaotic Dynamics on the Interval

The behavior of a periodic state under the iterations of the transition function of a given dynamical system is particularly simple. The abundance of such states may thus be regarded as how “well-behaved” is the system. And of course, topologically speaking, periodic states are “most abundant” if they form a dense collection in the set of all states. It is thus natural to ask if we can say something intelligent about those dynamical systems \((X, f)\) for which Per\((X, f)\) is dense in \(X\).

It is difficult to deal with this issue at a general level. But there is one very important case in which we get a startlingly clean answer to our query. It turns out that the periodic states of a topologically transitive dynamical system whose state space is a compact interval is, per force, dense. This is an extraordinary result, proved by the Ukrainian mathematician Oleksandr Sharkovsky in 1964, which attests to the unexpected power of topological transitivity. We prove this result here by means of an elegant argument due to Vellekoop and Berglund (1994).

Lemma 2.7. Let \( I \) be an interval and \( f \) a continuous self-map on \( I \). Let \( J \) be a subinterval of \( I \) such that \( J \cap \text{Per}(I, f) = \emptyset \), and take any \( x \in J \). If there exist
positive integers \(m\) and \(n\) such that \(m < n\) and \(\{f^m(x), f^n(x)\} \subseteq J\), then \(f^m(x)\) belongs to the interior of the interval between \(x\) and \(f^n(x)\).

**Proof.** Since \(x\) is in \(J\), it is not periodic under \(f\), and hence, \(x\) cannot be equal to \(f^m(x)\). Thus, either \(x < f^m(x)\) or \(f^m(x) < x\). We assume the former case here; the analysis in the latter case is analogous. Put \(g := f^m\). Then, by hypothesis, \(x < g(x)\). Next, we claim that \(g(x) < g^k(x)\) for every \(k = 2, 3, \ldots\). Suppose this is not the case, and let \(k\) be the smallest integer in \(\{2, 3, \ldots\}\) for which this inequality fails. Then, \(x < g^{k-1}(x)\) and \(g(x) \geq g^k(x)\). But then, for the map \(F := g^{k-1} - \text{id}_I\) on \(I\), we have \(F(x) > 0\) and \(F(g(x)) \leq 0\), so, by the Intermediate Value Theorem, \(g^{k-1}(z) = z\) for some \(z \in (x, g(x)]\). But then \(z\) is periodic under \(f\), so it cannot belong to \(J\), while, of course, \((x, g(x)) \subseteq J\), a contradiction. Conclusion: \(x < g^k(x)\) for every \(k \in \mathbb{N}\).

Now, to derive a contradiction, suppose that \(f^m(x)\) does not belong to the interior of the interval between \(x\) and \(f^n(x)\), that is, \(f^m(x) > f^n(x)\). For \(y := f^m(x)\), this inequality reads as \(y > f^{n-m}(y)\), so applying our previous argument, this time \(f^{n-m}\) playing the role of \(g\), we find that \(y > f^{(n-m)k}(y)\) for every \(k \in \mathbb{N}\).

Setting \(k\) as \(n - m\) in the conclusion of the first paragraph of the proof, and as \(m\) in that of the previous paragraph, we have \(x < f^{(n-m)m}(x)\) and \(y > f^{(n-m)m}(y)\). But then, for the map \(G := f^{(n-m)m} - \text{id}_I\) on \(I\), we have \(G(x) > 0\) and \(G(y) < 0\), so, by the Intermediate Value Theorem, \(f^{(n-m)m}(w) = w\) for some \(w\) in the interior of the interval between \(x\) and \(y\). Thus, \(z\) is periodic under \(f\), so it cannot belong to \(J\), while, of course, \((x, y) \subseteq J\), a contradiction.

We now use Lemma 2.7 to establish our main objective.

**Theorem 2.8.** (Sharkovsky) Let \(I\) be an interval and \(f\) a continuous self-map on \(I\). If \((I, f)\) is topologically transitive, then \(\text{Per}(I, f)\) is dense in \(I\).

**Proof.** To derive a contradiction, suppose there exists a nonempty open subinterval of \(I\), say, \(J := (a, b)\), such that \(J\) contains no point of \(I\) that is periodic under \(f\). But, by Lemma 2.3, there is an \(f\)-transitive state, say, \(x\), in \(I\). Then, \(f^m(x)\) belongs to \((a, b)\) for some \(m \in \mathbb{N}\). Since the orbit of \(x\) under \(f\) is dense in \(I\), the intersection of this set with the interval \((a, f^m(x))\) contains infinitely many points. Therefore, there is an integer \(n > m\) such that \(f^n(x)\) belongs to \((a, f^m(x))\). This contradicts Lemma 2.7.

Combining this theorem with Theorem 2.6, we obtain a far reaching result about chaotic dynamics on an interval. Apparently, any topologically transitive dynamical system whose state space is an interval is chaotic in the sense of Devaney.

**Corollary 2.9.** Let \(I\) be a compact interval and \(f\) a continuous self-map on \(I\). Then, \((I, f)\) is topologically transitive if, and only if, it is chaotic in the sense of Devaney.

**Exercises**
2.1. For any dynamical system \((X, f)\) and \(x \in \text{Per}(X, f)\), show that \((\text{orb}_f(x), f|_{\text{orb}_f(x)})\) is topologically transitive.

2.2. Let \((X, f)\) and \((Y, g)\) be two dynamical systems. Prove:
   a. If \((X, f) \cong (Y, g)\), then \((X, f)\) is topologically transitive iff so is \((Y, g)\);
   b. If \((Y, g)\) is a factor of \((X, f)\) – recall Exercise 1.7 – then topological transitivity of \((X, f)\) implies that of \((Y, g)\), but not conversely.

2.3. Let \((X, f)\) be the dynamical system introduced in Example 1.5 (where \(f\) is the doubling map). We have proved in Example 2.6 that this system is topologically transitive. Use part (c) of Exercise 1.7 and Example 2.6 to give an alternative proof of this fact.

2.4. a. Prove that the dynamical system \(([0, 1], f)\), where \(f\) is the tent map (Exercise 1.4) is topologically transitive, and conclude that this system is chaotic in the sense of Devaney.
   b. Prove that the dynamical system \(([0, 1], g)\), where \(g(x) := 4x(1-x)\), is chaotic in the sense of Devaney.

2.5.\(^H\) Prove: a dynamical system \((X, f)\) is topologically transitive iff for every nonempty open subsets \(O\) and \(U\) of \(X\), there exists a nonnegative integer \(k\) such that \(f^{-k}(O) \cap U \neq \emptyset\).

2.6. Let \((X, f)\) be a topologically transitive dynamical system such that \(X\) has at least one isolated point. Prove that \(X\) is finite and \(X = \text{orb}_f(x)\) for any \(x \in X\).

2.7.\(^H\) Let \((X, f)\) be a dynamical system and \(O\) and \(U\) two nonempty open sets in \(X\). Define \(N(O, U) := \{k \in \mathbb{N} : f^k(O) \cap U \neq \emptyset\}\). Prove: \((X, f)\) is topologically transitive iff \(N(O, U)\) is an infinite set.

2.8. (Birkhoff’s Transitivity Theorem) For any dynamical system \((X, f)\), prove:
   a. If \(X\) is a Polish metric space and \((X, f)\) is topologically transitive, then there is an \(f\)-invariant state in \(X\);
   b. If \(X\) is a separable normed linear space, and there is an \(f\)-transitive state in \(X\), then \((X, f)\) is topologically transitive;
   c. In either of the cases covered in (a) and (b), the set of all \(f\)-transitive states in \(X\) is dense in \(X\).

2.9. Let \((X, f)\) be the dynamical system such that \(X\) is second-countable and locally compact.
   a. Show that \((\text{cl}(\text{orb}_f(x)), f|_{\text{cl}(\text{orb}_f(x))})\) is topologically transitive for every \(x \in X\).
   b. Prove that there exists a partition \(\mathcal{Y}\) of \(X\) such that \((Y, f|_Y)\) is a topologically minimal dynamical system for each \(Y \in \mathcal{Y}\).
2.10. Let \((X, f)\) be a dynamical system. We say that a nonempty subset \(S\) of \(X\) is **invariantly connected** if there do not exist two nonempty disjoint open and \(f\)-invariant subsets \(O\) and \(U\) of \(X\) such that \(O \cup U = X\). Assume that \(f\) is open, and prove that \((X, f)\) is topologically transitive iff \(X\) is invariantly connected.

2.11. Let \((X, f)\) be a topologically transitive dynamical system with \(f\) being open. Prove that any real map \(f \circ \varphi\) on \(X\) with \(\varphi \circ f = \varphi\) is constant.

2.12. Let \((X, f)\) be a topologically transitive dynamical system with \(f\) being open and surjective. Suppose that there exist a real number \(\lambda \in \mathbb{R}\) and a nonzero \(F \in C_b(X)\) such that \(F(f(x)) = \lambda F(x)\) for every \(x \in X\). Prove that \(|\lambda| = 1\) and \(|F|\) is a constant function.

2.13. Let \((X, f)\) and \((Y, g)\) be two topologically conjugate dynamical systems where \(X\) and \(Y\) are compact metric spaces. Show that \((X, f)\) exhibits sensitive dependence on initial conditions iff \((Y, g)\) does so as well.

2.14. A dynamical system \((X, f)\) is said to be **topologically mixing**, if for every nonempty \(O, U \in \mathcal{O}_X\), there exists a nonnegative integer \(K\) such that \(f^k(O) \cap U \neq \emptyset\) for every integer \(k \geq K\). Prove that \((X, f)\) is topologically mixing iff for any strictly increasing sequence \((m_k)\) in \(\mathbb{N}\), and any nonempty \(O, U \in \mathcal{O}_X\), there is a \(k \in \mathbb{Z}_+\) such that \(f^{m_k}(O) \cap U \neq \emptyset\).

2.15. Let \((X, f)\) be a dynamical system such that \(X\) is a metric space. Show that if \((X, f)\) is topologically mixing, then it exhibits sensitive dependence on initial conditions.

2.16. Let \((X, f)\) and \((Y, g)\) be two dynamical systems, and define the self-map \(h\) on \(X \times Y\) by \(h(x, y) := (f(x), g(y))\). Then, \((X \times Y, h)\) is a dynamical system.

a. Prove that if \((X \times Y, h)\) is topologically transitive, then so are \((X, f)\) and \((Y, g)\).

b. Take any \(\alpha \in [0, 1]\), and consider the system \(([0, 1]^2, h_\alpha)\), where \(h_\alpha(x, y) := (f_\alpha(x), g_\alpha(y))\) — recall Example 1.4. Show that there is no \(h_\alpha\)-transitive state in \([0, 1]^2\), and conclude that the converse of what we have found in part (a) is false.

c. Prove that if \((X, f)\) is topologically transitive and \((Y, g)\) is topologically mixing, then \((X \times Y, h)\) is topologically transitive.

d. Prove that \((X \times Y, h)\) is topologically mixing iff so are \((X, f)\) and \((Y, g)\).

e. Use parts (b) and (c) to conclude that \(([0, 1], f_\alpha)\) is not topologically mixing for any irrational \(\alpha \in [0, 1]\). (Thus, topological transitivity does not imply topological mixing.)

2.17. (Grosse-Erdmann) Let \(X\) and \(Y\) be two topological spaces and \(\mathcal{F}\) a nonempty subset of \(C(X, Y)\). We say that \(\mathcal{F}\) is **universal** if there is a point \(x\) in \(X\) such that \(\{f(x) : f \in \mathcal{F}\}\) is dense in \(Y\), and denote the set of all such \(x\) by \(\mathbb{U}(\mathcal{F})\).
a. Assume that $X$ is a Baire space and $Y$ is second-countable, and prove: $\mathbb{U}(\mathcal{F})$ is a dense $G_\delta$-set in $X$ iff for every $(O, U) \in \mathcal{O}_X \times \mathcal{U}_X$, there exists an $f \in \mathcal{F}$ such that $f(O) \cap U \neq \emptyset$.

b. Assume that $X$ is a second-countable Baire space and $Y = X$, and prove: If $f(X)$ is dense in $Y$ for every $f \in \mathcal{F}$, and $f \circ g = g \circ f$ for every $f, g \in \mathcal{F}$, then either $\mathbb{U}(\mathcal{F}) = \emptyset$ or $\mathbb{U}(\mathcal{F})$ is dense in $X$.

2.4 Topological Minimality

Topologically Minimal Dynamical Systems

As we have seen above, the existence of a transitive state in a dynamical system has important implications with respect to the long-run behavior of the system. And in some special dynamical systems, all states behave in this manner. That is, there are systems that carry any given initial state across the entire state space (in the sense that the orbit of any state under the transitions of such a system is dense). Such dynamical systems bear a special name:

**Definition.** A dynamical system $(X, f)$ is said to be **topologically minimal** (or simply **minimal**) if every $x$ in $X$ is $f$-transitive.

Topologically minimal systems are referred to as such, because there is a sense in which such systems cannot be broken down to two or more subsystems.

**Proposition 2.10.** A dynamical system $(X, f)$ is topologically minimal if, and only if, the only closed $f$-invariant subset of $X$ is itself.

**Proof.** Let $(X, f)$ be minimal, and let $Y$ be a closed $f$-invariant subset of $X$ (which is, by definition of $f$-invariance, nonempty). Then, pick any $x$ in $Y$, and notice that $\text{cl}(\text{orb}_f(x)) \subseteq Y$. By the minimality of $(X, f)$, therefore, we have $X = Y$. Conversely, suppose that the only closed $f$-invariant subset of $X$ is itself. Since $\text{cl}(\text{orb}_f(x))$ is a closed $f$-invariant subset of $X$, then, $\text{orb}_f(x)$ is dense in $X$, for every $x \in X$.

As for examples, we note that topological transitivity and topological minimality coincide in the context of the Examples 2.2, 2.3, 2.4 and 2.5. In particular, irrational rotations on the circle yield topologically minimal dynamical systems. But, of course, a topologically transitive dynamical system need not be topologically minimal in general. Indeed, if a continuous self-map $f$ on a Hausdorff space $X$ has a fixed point, then $(X, f)$ is topologically...
minimal iff $|X| = 1$, while this system may well be topologically transitive. (Similarly, if $X$ is infinite, and there is at least one periodic state in $X$ under $f$, then this system cannot be minimal.) For instance, the transition maps of the topologically transitive systems we looked at in Example 2.6 (the doubling map) and Example 2.8 (shifts) have fixed points, and hence neither of these systems is topologically minimal.

**Isometric Topological Transitivity implies Minimality**

It turns out that the reason why topological transitivity and minimality coincide in the case of an irrational rotation stems from the fact that this system moves points around the state space in a distance-preserving manner. This is the content of the following result which provides a good source of topologically minimal systems.

**Proposition 2.11.** Let $(X, f)$ be a dynamical system such that $X$ is a complete and separable metric space and $f$ an isometry. If $(X, f)$ is topologically transitive, then it is minimal.

**Proof.** Assume that $(X, f)$ is topologically transitive. Since every complete and separable metric space is a second-countable Baire space, we may apply Lemma 2.3 to find an $f$-transitive state $x^*$ in $X$. Now, take any state $x$ in $X$, and note that for any $y \in X$ and $\varepsilon > 0$, there exist positive integers $k$ and $l$ such that $d(y, f^k(x^*)) < \varepsilon/2$ and $d(x, f^l(x^*)) < \varepsilon/2$, where $d$ stands for the metric of $X$. Without loss of generality, assume $k \geq l$, and note that

\[
\begin{align*}
  d(y, f^{k-l}(x)) &\leq d(y, f^k(x^*)) + d(f^k(x^*), f^{k-l}(x)) \\
  &= d(y, f^k(x^*)) + d(f^l(x^*), x) \\
  &< \varepsilon
\end{align*}
\]

where the equality follows from the fact that $f$ is an (invertible) isometry. In view of the arbitrary choice of $y$, this means that $\text{orb}_f(x)$ is dense in $X$. Conclusion: $(X, f)$ is minimal.

**Topological Minimality implies Surjectivity under Compactness**

Topological minimality is surely a demanding property. In fact, a dynamical system with a compact state space can never be minimal unless its transition function is an identification map.

**Lemma 2.12.** Let $(X, f)$ be a topologically minimal dynamical system such that $X$ is compact. Then, $f$ is a surjective identification map.
Proof. Surjectivity of $f$ is easily established by using Lemma 1.1. On the other hand, given that $X$ is a compact Hausdorff space, and $f$ is surjective, the Homeomorphism Theorem entails that $f$ is an identification map.

Topologically Minimal Dynamical Subsystems

Topological minimality is surely a demanding property. It is thus quite surprising that it is always met by some subsystem of any given dynamical system with a compact state space. We prove this fact next.

Theorem 2.13. (Birkhoff) Let $(X, f)$ be a dynamical system such that $X$ is compact. Then, there exists a closed and strongly $f$-invariant subspace $Y$ of $X$ such that $(Y, f|_Y)$ is a topologically minimal dynamical system.

Proof. Let $\mathcal{C}$ stand for the collection of all closed $f$-invariant subsets of $X$, and note that $\mathcal{C} \neq \emptyset$ because $X \in \mathcal{C}$. We wish to apply Zorn’s Lemma to the poset $(\mathcal{C}, \subseteq)$. To see that this poset is inductive, let $\mathcal{D}$ be a $\subseteq$-chain in $\mathcal{C}$. Then, by Proposition 1.1 of Chapter 7, $\bigcap \mathcal{D}$ is a nonempty closed subset of $X$. As it is plain that $f(\bigcap \mathcal{D}) \subseteq \bigcap \{f(D) : D \in \mathcal{D}\}$, we thus find that $\bigcap \mathcal{D} \in \mathcal{C}$. Conclusion: $(\mathcal{C}, \subseteq)$ is inductive. By Zorn’s Lemma, then, there is an $\subseteq$-maximal (that is, $\supseteq$-minimal) element, say, $Y$, of $\mathcal{C}$. Obviously, $(Y, f|_Y)$ is a dynamical system. Moreover, for every $x$ in $Y$, $\text{cl}(\text{orb}(f(x)))$ is a closed $f$-invariant subset of $Y$ – recall Example 1.8 – so this set is a member of $\mathcal{C}$. By $\supseteq$-minimality of $Y$ in $\mathcal{C}$, therefore, $\text{cl}(\text{orb}(f(x))) = Y$ for every $x \in Y$, that is, $(Y, f|_Y)$ is topologically minimal. Finally, since $Y$ is a closed subspace of $X$, it is compact, and hence, by Lemma 2.12, $f|_Y$ is surjective. This means that $Y$ is strongly $f$-invariant.

Exercises

2.17. For any dynamical system $(X, f)$ and $x \in \text{Per}(X, f)$, show that $(\text{orb}(f(x)), f|_{\text{orb}(f(x))})$ is topologically minimal.

2.18. Let $(X, f)$ and $(Y, g)$ be two dynamical systems. Prove:
   a. If $(X, f) \cong (Y, g)$, then $(X, f)$ is topologically minimal iff so is $(Y, g)$;
   b. If $(Y, g)$ is a factor of $(X, f)$ – recall Exercise 1.7 – then topological minimality of $(X, f)$ implies that of $(Y, g)$, but not conversely.

2.19. (Translations on the Torus) Take any $\alpha, \beta \in [0, 1)$ and let $(X, f_\alpha)$ and $(X, f_\beta)$ be two dynamical systems of the form introduced in Example 1.4. Define the self-map $f_{\alpha, \beta}$ on $X \times X$ by $f_{\alpha, \beta}(x_1, x_2) := (f_\alpha(x_1), f_\beta(x_2))$.
   a. Show that if $\alpha$ and $\beta$ are rational numbers, then every point in $X \times X$ is periodic under $f_{\alpha, \beta}$.
b. Show that if \( \alpha \) is an irrational number, then no point in \( X \times X \) is periodic under \( f_{\alpha,0} \), but unlike Example 1.4, \( (X \times X, f_{\alpha,0}) \) is not minimal.

c. Prove that \( (X \times X, f_{\alpha,\beta}) \) is a topologically transitive dynamical system if \( 1, \alpha, \beta \) are rationally independent (that is, for every integers \( a \) and \( b \), we have \( a\alpha + b\beta \in \mathbb{Z} \) iff \( a = b = 0 \)). Then, use Proposition 2.8 to conclude that this system is minimal.

2.20. For any dynamical system \( (X, f) \), we say that \( f \) is \textbf{irreducible} if it is surjective and \( f(S) \neq X \) for any proper closed subset \( S \) of \( X \). Assume \( X \) is compact, and show that if \( (X, f) \) is topologically minimal, then \( f \) is irreducible.

2.21. Let \( (X, f) \) be a topologically minimal dynamical system such that \( X \) is compact.

a. Show that \( f \) is open iff it is a homeomorphism.

b. Prove: For every nonempty \( O \in \mathcal{O}_X \), there is a positive integer \( m \) with \( O \cup f(O) \cup \cdots \cup f^m(O) = X \).

2.22. (Kinoshita) Let \( (X, f) \) be a dynamical system such that \( X \) is a compact and connected metric space. Show that either the set of all \( f \)-transitive points in \( X \) has empty interior or \( (X, f) \) is topologically minimal.

2.23. (Kolyada-Snoha-Trofimchuk) Let \( (X, f) \) be a topologically minimal dynamical system such that \( X \) is a compact metric space. Prove:

a. \( \{ x \in X : f^{-1}(x) \text{ is a singleton} \} \) is a dense \( G_\delta \)-set in \( X \);

b. Either \( f \) is a homeomorphism or (i) there is a subspace \( Y \) of \( X \) such that \( f|_Y \) is a homeomorphism from \( Y \) onto \( Y \), and (ii) \( (Y, f|_Y) \) is topologically minimal.

3  

Recurrence

3.1  \( \omega \)-Limit Sets

In the context of a dynamical system, one is particularly interested in those states \( x \) that tend to return to states that are close to \( x \). Such “recurrent” states are best studied by means of \( \omega \)-limit sets which we introduce next.

**Definition.** Let \( (X, f) \) be a dynamical system and \( S \) a nonempty subset of \( X \). The \textbf{\( \omega \)-limit set of \( S \) under \( f \)} is the set of all \( y \in X \) for which there exists a sequence \( (x_k) \in S^\infty \) and a strictly increasing sequence \( (m_k) \in \mathbb{N}^\infty \) such that \( f^{m_k}(x_k) \to y \). We denote the \( \omega \)-limit set of \( S \) under \( f \) as \( \omega_f(S) \), but if \( S \) is a singleton, say, \( S = \{x\} \), we write \( \omega_f(x) \) instead of \( \omega_f(\{x\}) \).

In the case where \( S \) is a singleton, say, \( S = \{x\} \), then, \( \omega_f(x) \) is none other than the collection of all subsequential limits of the trajectory \( (f^m(x)) \)
of $x$. It is this case that is particularly relevant for our purposes, but let us nevertheless make a few preliminary observations about $\omega$-limit sets in general.

**Lemma 3.1.** Let $(X, f)$ be a dynamical system, and $S$ a nonempty subset of $X$. Then, $\omega_f(S)$ is $f$-invariant, and we have

$$\omega_f(S) \subseteq \bigcap_{m=0}^{\infty} \text{cl} \left( \bigcup_{i=m}^{\infty} f^i(S) \right).$$

**Proof.** Take any $y \in X$ for which there exists an $(x_k) \in S^\infty$ and a strictly increasing $(m_k) \in \mathbb{N}^\infty$ such that $f^{m_k}(x_k) \to y$. Since $f$ is continuous, $f^{m_k+1}(x_k) \to f(y)$, and hence $f(y) \in \omega_f(S)$; this proves that $f(\omega_f(S)) \subseteq \omega_f(S)$. Now, pick any $m \in \mathbb{Z}_+$, and choose $K \in \mathbb{N}$ large enough that $m_k \geq m$ for every $k \geq K$. Then, trivially, $f^{m_k}(x_k) \in f^m(S) \cup f^{m+1}(S) \cup \cdots$ for every $k \geq K$, so $y \in \text{cl}(f^m(S) \cup f^{m+1}(S) \cup \cdots)$; this proves the second part of our claim.

When the state space at hand is first-countable, we can sharpen this observation nicely.

**Lemma 3.2.** Let $(X, f)$ be a dynamical system such that $X$ is first-countable, and let $S$ be a nonempty subset of $X$. Then, $\omega_f(S)$ is a closed $f$-invariant set, and we have

$$\omega_f(S) = \bigcap_{m=0}^{\infty} \text{cl} \left( \bigcup_{i=m}^{\infty} f^i(S) \right). \quad (5)$$

**Proof.** As the closedness of $\omega_f(S)$ follows from (5), and the $\subseteq$ part of (5) is established above, we only need to prove the $\supseteq$ part of (5). To this end, take any $y \in X$ such that $y \in \text{cl}(\bigcup_{i=m}^{\infty} f^i(S))$ for every $m \in \mathbb{Z}_+$. Let $B(y)$ be a countable local basis at $y$, which we enumerate as $\{B_1, B_2, \ldots\}$. Next, define $O_m := B_1 \cap \cdots \cap B_m$ for each $m \geq 1$, and note that $O_1 \supseteq O_2 \supseteq \cdots$. Since $O_1 \cap \bigcup_{i=1}^{\infty} f^i(S) \neq \emptyset$ (because this set contains $y$), there is an $m_1 \in \mathbb{N}$ and $x_1 \in S$ with $f^{m_1}(x_1) \in O_1$. Since, similarly, $O_2 \cap \bigcup_{i=m_1+1}^{\infty} f^i(S) \neq \emptyset$, there is an integer $m_2 \geq m_1 + 1$ and $x_2 \in S$ with $f^{m_2}(x_2) \in O_2$. Continuing this way, we find a strictly increasing sequence $(m_k) \in \mathbb{N}^\infty$ and a sequence $(x_{m_k})$ in $S$ such that $f^{m_k}(x_k) \in O_k$ for every $k \in \mathbb{N}$. But, for every $O \in O_X(y)$, there is a positive integer $K$ large enough so that $O_k \subseteq O$ for every $k \geq K$, and hence, $f^{m_k}(x_k) \in O$ for every $k \geq K$. This proves that $f^{m_k}(x_k) \to y$, whence $y \in \omega_f(S)$.  

33
Compactness of \(\omega\)-Limit Sets

In general, an \(\omega\)-limit set may well be empty. For instance, we have \(\omega_f(x) = \emptyset\) for every \(x \in \mathbb{R}\), where \(f\) is the self-map on \(\mathbb{R}\) given by \(f(x) := x + 1\). However, when \(X\) is compact, this issue does not arise.

**Lemma 3.3.** Let \((X, f)\) be a dynamical system such that \(X\) is first-countable and compact, and let \(S\) be a nonempty subset of \(X\). Then, \(\omega_f(S)\) is a nonempty, compact and strongly \(f\)-invariant set.

**Proof.** By Proposition 1.1 of Chapter 7, and Lemma 3.2, \(\omega_f(S) \neq \emptyset\). Moreover, since \(\omega_f(S)\) is closed in \(X\) (Lemma 3.2), it is compact. It remains to show that \(f(\omega_f(S)) \supseteq \omega_f(S)\). To this end, take any \(y \in \omega_f(S)\) so that there exists an \((x_k) \subseteq S\) and a strictly increasing sequence \((m_k) \subseteq \mathbb{N}\) with \(f^{m_k}(x_k) \to y\). Since \(X\) is first-countable and compact, it is sequentially compact – yes? – so the sequence \((f^{m_k-1}(x_k))\) has a subsequence that converges to some \(z\) in \(X\). By definition of \(\omega_f(S)\), we have \(z \in \omega_f(S)\). But, clearly, \(f(z) = y\), so we find \(y \in f(\omega_f(S))\), as we sought.

Connectedness of \(\omega\)-Limit Sets

Every periodic state belongs to its own \(\omega\)-limit set, that is, \(\text{orb}_f(x) = \omega_f(x)\) for every \(x \in \text{Per}(X, f)\). Thus, in general, \(\omega_f(x)\) is not connected. (For instance, in the context of a rational rotation of the circle, the \(\omega\)-limit set of any state is finite, and hence, disconnected.) However, while \(\omega_f(x)\) may be expressed as the union of two of its disjoint open subsets, these sets cannot be \(f\)-invariant, at least when \(X\) is a compact metric space. In other words, in that case, \(\omega_f(x)\) is invariantly connected – we have encountered this concept already in Exercise 2.10 – that is, \(\omega_f(x) \neq O \sqcup U\) for any disjoint open (or closed) \(f\)-invariant subsets \(O\) and \(U\) of \(\omega_f(x)\).

For proving this result, we will use the following technical lemma.

**Lemma 3.4.** Let \(S\) be a nonempty compact subset of a metric space \(X\) and \(f \in \mathcal{C}(X, X)\). Then, for every \(\varepsilon > 0\) there is a \(\delta > 0\) such that \(f(B(S, \delta)) \subseteq B(f(S), \varepsilon)\).

**Proof.** Exercise.

**Proposition 3.5.** (LaSalle) Let \((X, f)\) be a dynamical system such that \(X\) is a compact metric space. Then, \(\omega_f(x)\) is invariantly connected for any \(x \in X\).

\(^3\text{Reminder. For any nonempty subset } T \text{ of } X \text{ and any real number } \gamma > 0, B(T, \gamma) := \{x \in X : d(x, T) < \gamma\}, \text{ where } d \text{ is the metric of } X.\)
Proof. Take any \( x \in X \) and suppose \( \omega_f(x) = C \sqcup D \) for two disjoint closed \( f \)-invariant subsets \( C \) and \( D \) of \( \omega_f(x) \). By Lemma 3.3, \( \omega_f(x) \), and hence both \( C \) and \( D \), are compact in \( X \). By Proposition 1.11 of Chapter 7, therefore, there is an \( \varepsilon > 0 \) such that \( B(C, \varepsilon) \cap B(D, \varepsilon) = \emptyset \). In turn, by Lemma 3.4, and because \( f(C) \subseteq C \) and \( f(D) \subseteq D \), there is an \( \delta \) in \((0, \varepsilon)\) such that

\[
f(B(C, \delta)) \subseteq B(C, \varepsilon) \quad \text{and} \quad f(B(D, \delta)) \subseteq B(D, \varepsilon).
\] (6)

On the other hand, since \( X \) is compact, it is sequentially compact, so there is a positive integer \( M \) such that \( f^m(x) \in B(\omega_f(x), \delta) \) for each \( m \geq M \). (Otherwise, there would be a subsequence of \( f^m(x) \), say, \( (f^{m_k}(x)) \), that does not ever enter \( B(\omega_f(x), \delta) \); but this is impossible because \( (f^{m_k}(x)) \) must have a subsequential limit (as \( X \) is sequentially compact), and by definition of \( \omega_f(x) \), this limit belongs to \( \omega_f(x) \).) In particular,

\[
f^M(x) \in B(C, \delta) \quad \text{or} \quad f^M(x) \in B(D, \delta).
\]

Without loss of generality, suppose the former case holds. Then, by (6), \( f^{M+1}(x) \in B(C, \varepsilon) \) while \( f^{M+1}(x) \in B(\omega_f(x), \delta) \). But, given that \( B(C, \varepsilon) \cap B(D, \varepsilon) = \emptyset \), \( \omega_f(x) = C \sqcup D \), and \( \delta < \varepsilon \), we have \( B(C, \varepsilon) \cap B(\omega_f(x), \delta) \subseteq B(C, \delta) \), and hence we conclude that \( f^M(x) \in B(C, \delta) \). Continuing inductively, then, we find \( f^m(x) \in B(C, \delta) \) for every \( m \geq M \). But then \( D \) may contain no subsequential limit of \( (f^m(x)) \), so since \( D \subseteq \omega_f(x) \), we must have \( D = \emptyset \), a contradiction.

3.2 Recurrent and Almost Periodic States

Recurrent States

Let \((X, f)\) be a dynamical system. We say that a state \( x \) in \( X \) is recurrent under \( f \) if for every open neighborhood \( O \) of \( x \) in \( X \), there is a positive integer \( k \) large enough so that \( f^k(x) \in O \). Thus, if the system starts at a recurrent state then it eventually comes back arbitrarily close to that state. (If this happens once, then it happens infinitely often; see Exercise 3.3.) Obviously, every state that is periodic under \( f \) is recurrent under \( f \). But the converse is false, in general. For instance, in the context of an irrational rotation of the unit circle, every state is recurrent but none is periodic.

Example 3.1. All states in the dynamical systems given in Examples 1.1 and 1.2 are recurrent. On the other hand, in the context of the system \((\mathbb{R}, f)\), where \( f(x) := x + 1 \), no state in \( \mathbb{R} \) is recurrent under \( f \).
Example 3.2. Let $X$ be a complete metric space and $f$ a contraction on $X$. Then, no state in $X$, other than the fixed point $x^*$ of $f$, is recurrent under $f$. If $X$ contains only one point, this claim is trivial, so assume $|X| \geq 2$, and pick any $x \in X \setminus \{x^*\}$. Then, take any $U \in \mathcal{O}_X(x^*)$, and note that all but finitely many terms of the sequence $(f(x), f^2(x), \ldots)$ belong to $U$ (because $f^m(x) \to x^*$). But then, there is an open neighborhood $O$ of $x$ in $X$ such that $O \cap \operatorname{orb}_f(x) = \{x\}$. Thus, $x$ is not recurrent under $f$.

Evidently, a state $x$ in $X$ is periodic under $f$ iff $x \in \operatorname{orb}_f(x)$. When $X$ is first-countable, we can characterize the recurrent states analogously: $x$ is recurrent under $f$ iff $x \in \operatorname{cl}\{f(x), f^2(x), \ldots\}$. Moreover, in that case, a state is recurrent iff it is a subsequential limit of its trajectory:

Lemma 3.6. Let $(X, f)$ be a dynamical system such that $X$ is first-countable. Then, for any $x \in X$, the following are equivalent:

(a) $x$ is recurrent under $f$;  
(b) $x \in \operatorname{cl}\{f(x), f^2(x), \ldots\}$;  
(c) $x \in \omega_f(x)$.

Proof. Exercise.

Almost Periodic States

There are other ways of classifying states that “tend to return to somewhere close to themselves.” Given a dynamical system $(X, f)$, we say that a state $x$ in $X$ is **almost periodic under** $f$ if for every $O \in \mathcal{O}_X(x)$, there is a positive integer $M$ such that for every $m \in \mathbb{N}$, there is a $k \in \{m, \ldots, m + M\}$ with $f^k(x) \in O$. Obviously, such a state is recurrent under $f$. Indeed, almost periodic states are, in effect, *uniformly* recurrent states. We have:

$$\text{periodic state} \implies \text{almost periodic state} \implies \text{recurrent state}.$$  

None of these implications go the other way. In particular, the following example shows that a recurrent state need not be almost periodic. In turn, in Exercise 3.5, we show that an almost periodic state need not be periodic.

Example 3.3. Consider the dynamical system $(\{0, 1\}^\infty, \sigma)$, where $\sigma$ is the shift map on $\{0, 1\}^\infty$. Then, the sequence

$$x := (0, 1, 0, 0, 0, 1, 1, 0, 1, 0, 0, 0, \ldots, 1, 1, 1, \ldots),$$

whose construction we discussed in Example 2.8 (where $\omega_1 = 0$ and $\omega_2 = 1$). This sequence is called the **Morse sequence**, and it is recurrent under $\sigma$.
(because every $k$-block of this sequence (for any $k$) is repeated infinitely often throughout $x$. It is, however, not almost periodic under $\sigma$. Indeed, there are blocks in this sequence of the form $(0), (0,1,0), (0,1,1,0), \ldots, (0,1,\ldots,1,0)$, in which two 0s are separated by an arbitrarily long block of 1s. This implies that for $O := \{0\} \times \{0,1\}^\infty \times \{0,1\}^\infty \times \cdots$, there is no positive integer $M$ such that for every $m \in \mathbb{N}$, there is a $k \in \{m, \ldots, m + M\}$ with $f^k(x) \in O$.

There is a close connection between the properties of topological minimality and almost periodicity. In particular, when the state space of the system is first-countable and compact, we have the following characterization.

**Lemma 3.7.** Let $(X, f)$ be a dynamical system such that $X$ is first-countable and compact. Then, for any $x \in X$, the following are equivalent:

1. $x$ is almost periodic under $f$;
2. $(\text{cl}(\text{orb}_f(x))), f|_{\text{cl}(\text{orb}_f(x))}$ is topologically minimal;
3. There is a closed $f$-invariant subspace $Y$ of $X$ such that $x \in Y$ and $(Y, f|_Y)$ is topologically minimal.

**Proof.** $(a) \Rightarrow (b)$. Put $Y := \text{cl}(\text{orb}_f(x))$, and take any $y \in Y$. We are to show that $\text{orb}_f(y)$ is dense in $Y$. To do this, we will prove that $x \in \text{cl}(\text{orb}_f(y))$. This is enough, because, as $\text{cl}(\text{orb}_f(y))$ is closed and $f$-invariant, $x \in \text{cl}(\text{orb}_f(y))$ implies $Y \subseteq \text{cl}(\text{orb}_f(y))$, and we are done.

To derive a contradiction, suppose $x$ lies outside of $\text{cl}(\text{orb}_f(y))$. Then, by Corollary 1.7 of Chapter 7, there is an $O \in \mathcal{O}_X(x)$ such that

$$\text{cl}(O) \cap \text{cl}(\text{orb}_f(x)) = \emptyset. \quad (7)$$

Since $x$ is almost periodic under $f$, there is a positive integer $M$ such that

$$x \in \bigcup_{i=m}^{M+m} f^{-i}(O) \quad \text{for every } m = 1, 2, \ldots \quad (8)$$

In particular, $x \in \bigcup_{i=1}^{M+1} f^{-i}(O)$. On the other hand, applying (8) for $m = 2$ yields $f^k(x) \in O$ for some $k \in \{2, \ldots, M + 2\}$, and this implies $f(x) \in f^{-k+1}(O)$, and hence, $f(x) \in \bigcup_{i=1}^{M+1} f^{-i}(O)$. Continuing this way inductively, we find that $f^j(x) \in \bigcup_{i=1}^{M+1} f^{-i}(O)$ for every $j \in \mathbb{Z}_+$. It follows that

$$\text{orb}_f(x) \subseteq \bigcup_{i=1}^{M+1} f^{-i}(O) \subseteq \bigcup_{i=1}^{M+1} f^{-i}(\text{cl}(O)).$$
By continuity of $f$, the right-most set in this statement is closed, and hence

$$y \in \text{cl}(\text{orb}_f(x)) \subseteq \bigcup_{i=1}^{M+1} f^{-i}(\text{cl}(O)),$$

that is, $f^l(y) \in \text{cl}(O)$ for some $l \in \{1, \ldots, M+1\}$, but this contradicts (7).

(b) $\Rightarrow$ (c). This is trivial.

(c) $\Rightarrow$ (a). Without loss of generality, we may assume that $(X, f)$ is minimal in this case. Take any $x \in X$ and $O \in \mathcal{O}_X(x)$, and note that \{\text{orb}_f^i(y) : i = 0, 1, \ldots\} is an open cover of $X$. (Indeed, for any $y$ in $X$, denseness of orb$_f(y)$ in $X$ implies that there is a nonnegative integer $k$ such that $f^k(y) \cap O \neq \emptyset$, that is, $y \in f^{-k}(O)$.) Since $X$ is compact, therefore, there is a positive integer $M$ such that

$$X = \bigcup_{i=0}^{M} f^{-i}(O).$$

But for any $m \in \mathbb{N}$, this implies that $f^m(x) \in f^{-k}(O)$ for some $k \in \{0, \ldots, M\}$, and hence, $f^{m+k}(x) \in O$. In view of the arbitrary choice of $O$, we conclude that $x$ is almost periodic under $f$.

**Example 3.4.** Let $(\{0,1\}^\infty, \sigma)$ and $x$ be as in Example 3.3. As we have see in Example 2.8, the orbit of $x$ under $\sigma$ is dense in $\{0,1\}^\infty$. Thus, $(\text{cl}(\text{orb}_f(x)), f|_{\text{cl}(\text{orb}_f(x))})$ equals the entire system $(\{0,1\}^\infty, \sigma)$, which is not minimal (as it has fixed points). Thus, Lemma 3.7 provides another proof of the fact that the Morse sequence is not almost periodic under $\sigma$.

The following observation is an immediate consequence of Lemma 3.7.

**Corollary 3.8.** Let $(X, f)$ be a dynamical system such that $X$ is first-countable and compact. If $(X, f)$ is topologically minimal, then every state in $X$ is almost periodic under $f$.

Another corollary of Lemma 3.7 is the following famous theorem of George D. Birkhoff:

**The Birkhoff Recurrence Theorem.** Let $(X, f)$ be a dynamical system such that $X$ is first-countable and compact. Then, there is at least one state in $X$ that is almost periodic (hence recurrent) under $f$.

**Proof.** Apply Theorem 2.13 and Lemma 3.7.
Exercises

3.1. Let \((X, f)\) be a dynamical system such that \(X\) is first-countable and compact, and let \(B\) be a nonempty subset of \(X\) such that \(\omega_f(B) \subseteq B\). Show that \(\omega_f(B) = \bigcap_{m=0}^{\infty} f^m(B)\).

3.2. Prove Lemma 3.6.

3.3. Let \((X, f)\) be a dynamical system and \(x\) a recurrent state in \(X\) under \(f\). Show that \(x\) is \(\infty\)-recurrent in the sense that for every \(O \in \mathcal{O}_X(x)\), there exists a \(k \in \mathbb{N}\) such that \(f^k(O) \cap O \neq \emptyset\). The set of all such states in \(X\) is denoted by \(\Omega(X, f)\).

a. Show that the set of all wandering states in \(X\) under \(f\) is open and \(f\)-invariant. (\(\Omega(X, f)\) is thus closed and \(f\)-invariant.)

b. Show that \(\omega_f(x) \subseteq \Omega(X, f)\) for any \(x \in X\).

c. Prove: If \(X\) is first-countable, then every recurrent state in \(X\) under \(f\) is non-wandering under \(f\).

d. Prove: If \(X\) is compact, then \(\Omega(X, f) \neq \emptyset\). It follows that in every nonempty compact \(f\)-invariant set in \(X\) there is a state that is non-wandering under \(f\).

3.4. Let \((X, f)\) be a dynamical system. We say that a state \(y\) in \(X\) is non-wandering under \(f\) if for every \(O \in \mathcal{O}_X(y)\), there exists a \(k \in \mathbb{N}\) such that \(f^k(O) \cap O \neq \emptyset\). The set of all such states in \(X\) is denoted by \(\Omega(X, f)\).

\(\sigma\). Show that the set of all wandering states in \(X\) under \(f\) is open and \(f\)-invariant. (\(\Omega(X, f)\) is thus closed and \(f\)-invariant.)

\(\sigma\). Show that \(\omega_f(x) \subseteq \Omega(X, f)\) for any \(x \in X\).

c. Prove: If \(X\) is first-countable, then every recurrent state in \(X\) under \(f\) is non-wandering under \(f\).

d. Prove: If \(X\) is compact, then \(\Omega(X, f) \neq \emptyset\). It follows that in every nonempty compact \(f\)-invariant set in \(X\) there is a state that is non-wandering under \(f\).

3.5. Consider the sequence \((x_0, x_1, \ldots) \in \{0, 1\}^\infty\) defined, recursively, as follows: \(x_0 = 0\), \(x_{2m} = x_m\) and \(x_{2m+1} = 1 - x_m\), for every \(m = 0, 1, \ldots\). (Alternatively, we can describe this sequence as follows: Set \(x_0 = 0\); negate this “1-block” to get 1, and adjoin this to the previous “1-block” to get \(0, 1\); now negate this “2-block” (pointwise) to get \(1, 0\), and adjoin the result to the previous “2-block” to get \(0, 1, 1, 0\); now negate this “4-block” (pointwise) to get \(1, 0, 0, 1\), and adjoin the result to the previous “4-block” to get \(0, 1, 1, 0, 1, 0, 1\); continue this way.) This sequence is known as the Thue-Morse sequence. (Fun Fact. The \(m\)th term of this sequence is 1 if the number of 1s in the binary expansion of \(m\) is odd, and it is 0 otherwise.) Prove that the Thue-Morse sequence is almost periodic under the shift map \(\sigma\) on \(\{0, 1\}^\infty\), but it is not periodic under \(\sigma\).

4 Monotone Dynamical Systems

4.1 Ordered Topological Spaces

Ordered Topological Spaces

Let \((X, \succ)\) be a poset where \(X\) is a topological space.\(^4\) We say that \((X, \succ)\)

\(^4\)Reminder. Recall that \(\succ\) stands for the asymmetric part of the partial order \(\succ\), that is, \(x \succ y\) iff \(x \succ y\) and \(x \neq y\). Also recall that when we write \(S \succ T\), what we mean is \(x \succ y\) for every \((x, y) \in S \times T\). In turn, \(x \succ T\) means \(\{x\} \succ T\).
is an ordered topological space if \( \succ \) is a closed subset of \( X \times X \). (That is, for any two nets \((x_\alpha)\) and \((y_\alpha)\) with the same index poset, if \( x_\alpha \succ y_\alpha \) for each \( \alpha \), and \( x_\alpha \to x \) and \( y_\alpha \to y \) for some \( x, y \in X \), then \( x \succ y \).) It is worth noting that any such space is Hausdorff.

**Proposition 4.1.** (Eilenberg) Every ordered topological space is Hausdorff.

**Proof.** Let \((X, \succ)\) be an ordered topological space. We wish to show that \( \Delta_X \) is a closed subset of \( X \times X \). (Recall Proposition 1.2 of Chapter 5.) To this end, take any distinct \( x \) and \( y \) in \( X \); and note that both \( x \succ y \) and \( y \succ x \) cannot hold (because \( \succ \) is antisymmetric). Say, \( y \succ x \) is false, that is, \((y,x)\) does not belong to \( \succ \). Then, as \( \succ \) is closed, there is an open subset \( O \) of \( X \) such that \((y,x)\in O \) and \( O \cap \succ = \emptyset \). Since \( \Delta_X \) is contained in \( \succ \) (because \( \succ \) is reflexive), therefore, \( \Delta_X \cap \succ = \emptyset \), which proves that \((y,x)\) lies in the interior of \( \{(x,y) \in X \times X : x \neq y\} \). In view of the arbitrary choice of \( x \) and \( y \), then, we may conclude that \( \Delta_X \) is closed.

The theory of ordered topological spaces is fairly rich, but we will need only the following result from this theory – this is a generalization of the Weierstrass Maximum Value Theorem.

**Theorem 4.2.** (Wallace) Let \((X, \succ)\) be an ordered topological space. Every nonempty compact subset \( S \) of \( X \) contains a maximal element with respect to \( \succ \).

**Proof.** We begin by noting that, by the Hausdorff Maximal Principle, there exists a maximal \( \succ \)-chain in \( X \), say, \( S \). Next, we claim that \( \{x^\uparrow : x \in S\} \) has the finite intersection property, where \( x^\uparrow := \{y \in X : y \succ x\} \) for any \( x \in X \). Indeed, suppose \( T \) is a nonempty finite subset of \( S \). As \( T \) is a finite \( \succ \)-chain in \( X \), it has a maximum element, say, \( x^* \), with respect to the partial order \( \succ \). Clearly, \( x^* \) belongs to \( x^\uparrow \) for each \( x \in T \), that is, \( \bigcap \{x^\uparrow : x \in T\} \neq \emptyset \).

Now, it is plain that closedness of \( \succ \) entails that any set of the form \( x^\uparrow \) is closed in \( X \). Therefore, \( \{x^\uparrow : x \in S\} \) is a collection of closed subsets of \( X \) with the finite intersection property. As \( X \) is compact, therefore, \( \bigcap \{x^\uparrow : x \in S\} \) is nonempty (Proposition 1.1). It follows that there exists an element \( x^* \) in \( X \) such that \( x^* \succ S \). But then \( x^* \) must be maximal in \( X \) with respect to \( \succ \). For, otherwise, there would exist an element \( y \) of \( X \setminus S \) with \( y \succ x^* \), and hence \( S \cup \{y\} \) would be a \( \succ \)-chain in \( X \), contradicting the maximality of \( S \).

**Corollary 4.3.** Let \((X, \succ)\) be an ordered topological space, and \( S \) a nonempty relatively compact subset of \( X \). Suppose that \( X \) is first-countable
and \( \succeq \) is total on \( S \). Then, it is total on \( \text{cl}(S) \) and there is an \( x^* \) in \( \text{cl}(S) \) such that \( x^* \succeq \text{cl}(S) \).

**Proof.** Take any points \( x \) and \( y \) in \( \text{cl}(S) \). By Proposition 2.2 of Chapter 4, we have \( x_m \to x \) and \( y_m \to y \) for some sequences \( (x_m) \) and \( (y_m) \) in \( S \). By totality of \( \succeq \) on \( S \), there must exist a strictly increasing sequence \( (m_k) \) of positive integers such that either \( x_{m_k} \succeq y_{m_k} \) for each \( k \in \mathbb{N} \) or \( y_{m_k} \succeq x_{m_k} \) for each \( k \in \mathbb{N} \). Letting \( k \uparrow \infty \), and recalling that \( \succeq \) is closed in \( X \times X \), we then find that either \( x \succeq y \) or \( y \succeq x \). Since \( x \) and \( y \) were arbitrarily chosen in \( \text{cl}(S) \), we may thus conclude that \( \succeq \) is total on \( \text{cl}(S) \). The second part of our assertion follows from combining this fact with Lemma 4.2.

**Corollary 4.4.** Let \((X, \succeq)\) be an ordered topological space and \((x_m)\) a sequence in \( X \) such that \( \text{cl}\{x_1, x_2, \ldots\} \) is compact. Suppose that \( X \) is first-countable and \( \cdots \succeq x_2 \succeq x_1 \). Then, \((x_m)\) converges and \( \lim x_m \succeq \text{cl}\{x_1, x_2, \ldots\} \).

**Proof.** Put \( S := \{x_1, x_2, \ldots\} \). If \( S \) is finite, our claim is trivially true, so assume otherwise. By Corollary 4.3, there is an \( x^* \) in \( \text{cl}(S) \) with \( x^* \succeq \text{cl}(S) \). Note that \( x^* \) cannot be in \( S \), because if \( x^* = x_k \) for some \( k \), then \( x^* = x_l \) for all \( l \geq k \), which means that \( S \) is a finite set. It thus follows from Proposition 2.2 of Chapter 4 that there is a subsequence \((x_{m_k})\) of \((x_m)\) with \( x_{m_k} \to x^* \). In fact, \( \lim x_m = x^* \). To see this, take any subsequence \((x_{n_k})\) of \((x_m)\). Then, for every \( k \), there is a positive integer \( l(k) \) such that \( x^* \succeq x_{n_{k(l)}} \succeq x_{m_k} \). But then letting \( k \uparrow \infty \), and recalling that \( \succeq \) is closed in \( X \times X \), we find \( x_{n_{k(l)}} \to x^* \). That is, every subsequence of \((x_m)\) has itself a subsequence that converges to \( x^* \). This means that \( x_m \to x^* \).

**Strongly Ordered Topological Spaces**

Let \((X, \succeq)\) be an ordered topological space. In what follows, we denote the interior of \( \succeq \) in \( X \times X \) by \( \succ^0 \). In other words, \( \succ^0 \) is the partial order on \( X \) defined by \( x \succ^0 y \) iff \( O \succ U \) for some \( O \in \mathcal{O}_X(x) \) and \( U \in \mathcal{O}_X(y) \). (It is plain that \( \succ^0 \subseteq \succ \), but the converse containment often fails.) We say that \((X, \succeq)\) is a **strongly ordered topological space** if for every \( z \) in \( X \) and every \( O \in \mathcal{O}_X(z) \), there exist points \( y \) and \( z \) in \( O \) such that \( x \succ^0 z \succ^0 \).

Every subspace of an ordered topological space is itself an ordered topological space relative to the induced partial order. This not true for strongly ordered topological spaces in general. However, if \((X, \succeq)\) is a strongly ordered topological space, and \( Y \) is an open subset of \( X \), then \((Y, \succeq \cap (Y \times Y))\) is a strongly ordered topological space.
**Example 4.1.** Take any positive integer \( n \), and consider \( \mathbb{R}^n \) as ordered by the usual componentwise ordering \( \geq \). (That is, \( x \geq y \) iff \( x_i \geq y_i \) for each \( i = 1, \ldots, n \).) Then, \( x > y \) iff \( x_i \geq y_i \) for each \( i \) with at least one of these inequalities holding strictly, and \( x >^o y \) iff \( x_i > y_i \) for each \( i = 1, \ldots, n \).

We always think of \( \mathbb{R}^n \) as ordered by \( \geq \), unless otherwise is explicitly stated. Viewed this way, \( \mathbb{R}^n \) is a strongly ordered topological space. By contrast, endowing \( \mathbb{R}^n_+ \) with \( \geq \) yields an ordered topological space which is not strongly ordered.

**Example 4.2.** Define the binary relation \( \geq \) on \( C[0,1] \) by \( f \geq g \) iff \( f(t) \geq g(t) \) for every \( t \in [0,1] \). Then, \( \geq \) is a partial order on \( C[0,1] \), and endowing \( C[0,1] \) with this ordering yields an ordered topological space. Besides, for every continuous real map \( f \) on \( [0,1] \) and \( \varepsilon > 0 \), we have \( f + \frac{\varepsilon}{2} >^o f >^o f - \frac{\varepsilon}{2} \). This observation readily implies that \( C[0,1] \) is a strongly ordered topological space.

The following example generalizes both of the previous examples.

**Example 4.3.** Let \( X \) be a Banach space, and \( C \) a closed and convex cone in \( X \). (That is, \( C \) is a nonempty closed subset of \( X \) such that \( C \cap -C = \{0\} \), \( C + C \subseteq C \) and \( \lambda C \subseteq C \) for every \( \lambda \geq 0 \).) We define the binary relation \( \geq_C \) on \( X \) by \( x \geq_C y \) iff \( x - y \in C \). It is readily checked that \((X, \geq_C)\) is an ordered topological space. (In fact, the two examples we looked at above arise in exactly this way. In Example 4.1, \( C \) was \( \mathbb{R}^n_+ \), and in Example 4.2, it was \( \{ h \in C[0,1] : h \geq 0 \} \).) But \((X, \geq_C)\) may or may not be strongly ordered.

**Order-Preserving Self-Maps**

Let \((X, \geq)\) be an ordered topological space, and \( f \) a self-map on \( X \). We say that \( f \) is order-preserving (or monotonic) if

\[ x \geq y \quad \text{implies} \quad f(x) \geq f(y) \]

for every \( x, y \in X \), and strongly order-preserving (or strongly monotone) if

\[ x > y \quad \text{implies} \quad f(x) >^o f(y) \]

for every \( x, y \in X \).

**Example 4.4.** Consider \( \mathbb{R}^n \) as an ordered topological space as in Example 4.1. Then, a \( C^1 \)-map from \( \mathbb{R}^n \) into \( \mathbb{R}^n \) with a nonnegative Jacobian is order-preserving. Put more precisely, if \( f \) is a continuously differentiable self-map
on \( \mathbb{R}^n \) such that \( \frac{\partial f_i}{\partial x_j} \geq 0 \) for every \( i, j = 1, \ldots, n \), then \( f \) is order-preserving. If all inequalities here hold strictly, then \( f \) is strongly order-preserving.

**Example 4.5.** Consider \( C[0, 1] \) as an ordered topological space as in Example 4.2, and define the self-map \( F \) on \( C[0, 1] \) by

\[
F(f)(x) := \int_0^x f(t) \, dt \quad \text{for every } x \in [0, 1].
\]

Then, \( f \) is strongly order-preserving.

### 4.2 Monotone Dynamical Systems

#### Periodic Orbits in Monotone Dynamical Systems

A (strongly) monotone dynamical system is an ordered triplet \((X, \succ, f)\) such that \((X, \succ)\) is a (strongly) ordered topological space and \( f \) is a continuous and order-preserving self-map on \( X \). The periodic orbits in the context of such a dynamical system exhibit remarkable properties. A basic observation in this regard is that no two distinct points in such an orbit are ordered.

**Proposition 4.5.** Let \((X, \succ, f)\) be a monotone dynamical system. Then, for any \( x \in \text{Per}(X, f) \), no two elements of the orbit of \( x \) under \( f \) are ordered by \( \succ \).

**Proof.** Take an \( x \in \text{Per}(X, f) \), and assume that there is a positive integer \( k \) such that \( x \succ f^k(x) \). Since \( f^k \) is order-preserving, we then have \( x \succ f^k(x) \succ f^{2k}(x) \succ \cdots \). Not all \( \succ \) in this expression may hold as \( \succ \), for otherwise \( \text{orb}_f(x) \) would be finite, contradicting \( x \) being periodic under \( f \). So, there is a positive integer \( l \) such that \( f^{lk}(x) = f^{lk+k}(x) \). But this means that \( k \) must be a multiple of the period of \( x \) under \( f \), which, in turn, implies that \( x = f^k(x) \), a contradiction. Conclusion: \( x \succ f^k(x) \) does not hold for any \( k \in \mathbb{N} \). An analogous reasoning shows that \( f^k(x) \succ x \) cannot hold either, so \( x \) and \( f^k(x) \) are not ordered by \( \succ \), for any \( k \in \mathbb{N} \). This entails that no two elements of \( \text{orb}_f(x) \) are ordered \( \succ \).

Our next result provides a very nice criterion for the \( \omega \)-limit set of a state in a monotone dynamical system to be a periodic orbit (and hence a finite set).

**Lemma 4.6.** (Hirsch) Let \((X, \succ, f)\) be a monotone dynamical system, and \( x \) a state in \( X \) with a relatively compact orbit under \( f \). Suppose \( X \) is first-countable, and \( f^m(x) \succ x \) for some positive integer \( m \). Then, there is an
$x^* \in X$ such that the period of $x^*$ under $f$ is at most $m$, and $\omega_f(x) = \text{orb}_f(x^*)$. In particular, no two elements of $\omega_f(x)$ are ordered by $\preceq$.

**Proof.** Let us first consider the case where $m = 1$. Then, $\cdots \supseteq f^2(x) \supseteq f(x) \supseteq x$ so, by Corollary 4.4, $f^m(x) \to x^*$ for some $x^* \in X$ with $x^* \supseteq \text{cl}(S)$, where $S := \text{orb}_f(x)$. Note that $x^*$ is a fixed point of $f$ (so the period of $x^*$ under $f$ is 1), because $f(x^*) = f(\lim f^m(x)) = \lim f^{m+1}(x) = x^*$ by continuity of $f$. Moreover, $x^* \in \text{cl}\{f^m(x), f^{m+1}(x), \ldots\}$ for every $m \in \mathbb{N}$, so $x^* \in \omega_f(x)$ by Lemma 3.2. Conversely, take any $y$ in $\omega_f(x)$ so that there is a strictly increasing sequence $(m_k)_{k \in \mathbb{N}}$ in $\mathbb{N}$ with $f^{m_k}(x) \to y$. For every $l \in \mathbb{N}$, let $m(l)$ be the first integer in $\{m_1, m_2, \ldots\}$ with $m(l) \geq l$. Then, $f^{m(l)} \supseteq f^l(x)$, and letting $l \uparrow \infty$ yields $y \supseteq x^*$ (since $\supseteq$ is closed). As $y \in \text{cl}(S)$, this is possible only if $y = x^*$. Thus: $\omega_f(x) = \{x^*\}$.

Suppose now that $m \geq 2$. Applying the previous argument to the map $f^m$, we find that $f^{mk}(x) \to x^*$ (as $k \uparrow \infty$) for some $x^* \in X$ with $f^m(x^*) = x^*$. (Then, the period of $x^*$ under $f$ is at most $m$.) Besides, obviously, $x^* \in \omega_f(x)$, and by continuity of $f$, we have $f'(x^*) = \lim f^{mk+r}(x^*) \in \omega_f(x)$ for every $r = 1, \ldots, m - 1$. Thus, $\{x^*, f(x^*), \ldots, f^{m-1}(x^*)\} \subseteq \omega_f(x)$. Conversely, let $y$ be any element of $\omega_f(x)$ so that there is a strictly increasing sequence $(m_k)_{k \in \mathbb{N}}$ in $\mathbb{N}$ with $f^{m_k}(x) \to y$. Clearly, by the division algorithm, every $m_k$ can be written as $l_km + r_k$ for some $l_k \in \mathbb{Z}_+$ and $r_k \in \{0, \ldots, m-1\}$. But then there exists an $r \in \{0, \ldots, m-1\}$ such that infinitely many $m_k$s can be written as $l_km + r$. It follows that there is a strictly increasing sequence $(l_k)$ of positive integers such that $(l_k, m + r)$ is a subsequence of $(m_k)$. Consequently,

$$y = \lim_{k \to \infty} f^{mk}(x) = f^r \left( \lim_{n \to \infty} f^{lk_n m}(x) \right) = f^r(x^*).$$

Conclusion: $\omega_f(x) = \{x^*, f(x^*), \ldots, f^{m-1}(x^*)\}$.

The second part of Lemma 4.6 follows from combining its first part with Proposition 4.5.

**Attractors and Traps in Dynamical Systems**

Let $(X, f)$ be a dynamical system. We say that a subset $S$ of $X$ attracts a state $x$ in $X$ under $f$ if $x$ has a relatively compact orbit under $f$ and $\omega_f(x) \subseteq S$. Thus, if $S$ attracts $x$, we understand that $x$ has a “small orbit” and no subsequential limit of the trajectory of $x$ under $f$ may escape $S$. We say that $S$ attracts a subset $T$ of $X$ if it attracts $x$ for every $x \in T$.

A subset $S$ of $X$ is is said to be an $f$-attractor if $S$ is a compact $f$-invariant set that attracts an open set in $X$ that contains $S$. In turn, the
union of all open sets that \( S \) attracts is called the **basin of attraction** for \( S \), and is denoted by \( B_f(S) \). (Evidently, \( B_f(S) \) is an open \( f \)-invariant superset of \( S \). Moreover, \( \text{orb}_f(x) \) is relatively compact and \( \omega_f(x) \subseteq S \), for every \( x \in B_f(S) \).) Intuitively speaking, an \( f \)-attractor \( S \) is a "small" set (in the sense of being compact) that contains all limit points of the trajectories of all states in an open neighborhood of \( S \). The largest such neighborhood is the basis of attraction for \( S \).

We say that a subset \( S \) of \( X \) is an \( f \)-**trap** if it is a periodic orbit – that is, \( S = \text{orb}_f(x) \) for some \( x \in \text{Per}(X, f) \) – and it attracts an open set. In general, being an \( f \)-trap is extremely demanding. After all, such a set is finite, but it is supposed to contain all limit points of the trajectories of all states in an open set.

**Hirsch’s Theorem**

It is quite remarkable that in the context of strongly monotone dynamical systems we can find traps in any attractor, that is, in the context of such a system, every attractor contains a periodic orbit that attracts the entirety of an open set. This impressive observation, made by Morris Hirsch in 1985, has catapulted a large amount of work on monotone dynamics. (See, Hirsch and Smith (2005) for a nice survey of the related literature.) We conclude our introduction here with this theorem.

**Hirsch’s Theorem.** Let \((X, \succ, f)\) be a strongly monotone dynamical system such that \( X \) is first-countable. Then, there is an \( f \)-trap in every \( f \)-attractor of \( X \).

Before moving on to the proof of this theorem, recall that, for any dynamical system \((X, f)\), a state \( x \) in \( X \) is said to be **non-wondering under** \( f \) if for every \( O \in \mathcal{O}_X(x) \), there is a positive integer \( k \) such that \( \Omega^k(O) \cap O \neq \emptyset \); we denote the set of all non-wondering states in \( X \) under \( f \) by \( \Omega(X, f) \). We have worked with this concept a bit in Exercise 3.4 where we have seen that \( \Omega(X, f) \) is a closed and \( f \)-invariant subset of \( X \) that contains \( \omega_f(x) \) for each \( x \in X \). In the case of monotone dynamical systems, we can say a bit more.

**Lemma 4.7.** Let \((X, \succ, f)\) be a monotone dynamical system, and \( S \) a nonempty compact \( f \)-invariant subset of \( X \). Then, there is a maximal element of \( S \cap \Omega(X, f) \) with respect to \( \succ \).

**Proof.** By Exercise 3.4, \( S \cap \Omega(X, f) \) is a nonempty compact subset of \( X \). Apply Theorem 4.2.
We are now ready for:

**Proof of Hirsch’s Theorem.**\(^5\) Let \(S\) be an \(f\)-attractor in \(X\) so that there is an \(O \in \mathcal{O}_X\) such that \(S \subseteq O\) and \(S\) attracts \(O\) under \(f\). Besides, by Lemma 4.7, there is a \(\succ\)-maximal element \(z\) of \(S \cap \Omega(X, f)\). Since \((X, \succ)\) is a strongly ordered topological space, there is an \(x \in O\) with \(x \succ z\). Moreover, since \(z\) is non-wondering under \(f\), and \(X\) is first-countable, there exist a sequence \((x_m)\) in \(X\) and a strictly increasing sequence \((k_m)\) in \(\mathbb{N}\) such that \(x_m \to z\) and \(f^{k_m}(x_m) \to z\). Clearly, there is an integer \(M\) such that \(x \succ x_m\) for all \(m \geq M\). Besides, \(\text{cl}(\text{orb}_f(x))\) is compact (because \(S\) attracts \(x\) under \(f\)) and contains \(f^{k_m}(x)\) for each \(m\). It follows that \((f^{k_m}(x))\) has a convergent subsequence. Relabelling if necessary, we denote this subsequence also by \((f^{k_m}(x))\), and write \(y\) for its limit. Notice that, since \(f\) is order-preserving, we have \(f^{k_m}(x) \succ f^{k_m}(x_m)\) for all \(m \in \mathbb{N}\) with \(k_m \geq M\). As \(\succ\) is closed in \(X \times X\), therefore, we have \(y \succ z\). But \(\omega_f(x) \subseteq S\) (because \(S\) attracts \(x\) under \(f\)) while \(\omega_f(x) \subseteq \Omega(X, f)\) by Exercise 3.4. As \(y \in \omega_f(x)\), therefore, \(y \in S \cap \Omega(X, f)\). But in view of the maximality of \(z\), and because \(y \succ z\), this can happen iff \(y = z\). Since \(x \succ z\) and \(f^{k_m}(x) \to y = z\), therefore, it follows that \(x \succ f^k(x)\) for some positive integer \(k\). We may then apply Lemma 4.6 to conclude that \(\omega_f(x)\) is a periodic orbit that contains \(z\).

We now know that \(\omega_f(x)\) is a periodic orbit that contains \(z\) for every \(x\) in \(U := \{x \in O : x \succ z\}\). That is, for every \(x \in U\), there is a \(w_x \in \text{Per}(X, f)\) such that \(z \in \omega_f(x) = \text{orb}_f(w_x)\). Since \(z \in \text{orb}_f(w_x)\) for every \(x \in U\), we must have \(\text{orb}_f(w_x) = \text{orb}_f(w_{x'})\) for every \(x, x' \in U\). Therefore, for any fixed \(x \in U\), we have \(\omega_f(x') = \omega_f(x) = \text{orb}_f(w_x)\) for every \(x' \in U\). Moreover, every \(x' \in U\) has a relatively compact orbit under \(f\) because any such \(x'\) lies in \(U\) (as \(U \subseteq O\)) and \(S\) attracts \(O\) under \(f\). Thus: \(\text{orb}_f(w_x)\) is a periodic orbit that attracts \(U\).

We conclude by noting that the proof above actually shows that \(\text{orb}_f(z)\) is an \(f\)-trap. Thus, apparently, in a strongly monotonic dynamical system with a first-countable state space, the maximal non-wondering states in an attractor have periodic orbits that attract nonempty open sets.

**Exercises**

4.1. Let \((X, \succ, f)\) be a monotone dynamical system. Prove that for any \(x \in X\), no two elements of \(\omega_f(x)\) are ordered by \(\succ\).

\(^5\)The proof we give below is not the one given originally in Hirsch (1985) which seems to have a gap. Our argument rather follows the one given in Hirsch and Smith (2005).
4.2. Let $(X, \succ, f)$ be a strongly monotone dynamical system. Prove that for any $x \in X$, no two elements of $\omega_f(x)$ are ordered by $\succ$.

4.3. (Hirsch) Let $(X, \succ, f)$ be a strongly monotone dynamical system, and $S$ an $f$-attractor in $X$. Assume that the basin of attraction for $S$ is a $\lor$-semilattice, that is, for every $x, y \in B_f(S)$, there is a least upper bound for \{x, y\} (with respect to $\succ$) in $B_f(S)$. Prove that every maximal element $z$ of $S \cap \Omega(X, f)$ is a fixed point of $f$ and \{z\} is an $f$-trap.

5 Linear Dynamical Systems

We now consider what would happen if the state space of a dynamical system is a normed linear space and the transition function of the system is linear. At first sight, one might think that such a system would not have interesting dynamics; in particular, it seems like it cannot possibly exhibit chaotic behavior. And this is true when the state space of the system is finite-dimensional. However, there are quite a number of surprises in that regard when the state space is infinite-dimensional. Moreover, such dynamical systems arise in operator theory in all sorts of nontrivial ways, so it would be a mistake to think of them as uninteresting from the dynamical point of view. We thus provide a brief introduction to the theory of such systems here, using what we have learned so far about nonlinear systems.

Linear Dynamical Systems and Hypercyclicity

**Definition.** A linear dynamical system is an ordered pair $(X, f)$ where $X$ is a separable Banach space and $f$ a continuous linear self-map on $X$. In turn, if there is at least one $f$-transitive state in $X$, we say that $f$ is hypercyclic.

**Note.** In operator theory, a continuous linear self-map $f$ on a normed linear space $X$ is called cyclic if the linear span of $\text{orb}_f(x)$ is dense in $X$ for at least one $x \in X$. This justifies the terminology above – $f$ is hypercyclic if $\text{orb}_f(x)$ itself is dense in $X$ for at least one $x \in X$. (The terminology is due to Bernard Beauzamy.)

A separable Banach space is a second-countable Baire space, and it does not have any isolated points. Thus, as an immediate consequence of Theorem 2.4, we see here that the notion of hypercyclicity coincides with that of topological transitivity.

**Proposition 5.1.** (Birkhoff) A linear dynamical system $(X, f)$ is topologically transitive if, and only if, $f$ is hypercyclic.
Hypercyclicity in Finite-Dimensional Spaces

All goes according to intuition when $X$ is a finite-dimensional normed linear space. For the moment, we prove this for the case where $X = \mathbb{R}^n$.

**Proposition 5.2.** For any positive integer $n$ and linear dynamical system $(\mathbb{R}^n, f)$, the map $f$ is not hypercyclic.

A key observation toward the proof of this result is the following:

**Lemma 5.3** Let $(X, f)$ be a linear dynamical system such that the dimension of $X$ is $n \in \mathbb{N}$. If $x$ is an $f$-transitive state in $X$, then $\{x, f(x), ..., f^{n-1}(x)\}$ is a basis for $X$.

**Proof.** Suppose $x \in X$ is $f$-transitive. Then, $\text{orb}_f(x)$ cannot be finite, which means that all of its elements are distinct. Consequently, it is enough to prove that the vectors $x, f(x), ..., f^{n-1}(x)$ are linearly independent. Suppose this is not true. Then, there is an integer $k \in \{1, ..., n - 1\}$ such that $f^k(x)$ is in the linear span $Y$ of $\{x, f(x), ..., f^{k-1}(x)\}$, that is,

$$f^k(x) = \sum_{i=0}^{k-1} \alpha_i f^i(x)$$

for some real numbers $\alpha_1, ..., \alpha_{k-1}$. But then

$$f^{k+1}(x) = \sum_{i=0}^{k-1} \alpha_i f^{i+1}(x) = \sum_{i=0}^{k-2} \alpha_i f^{i+1}(x) + \alpha_{k-1} f^k(x) \in Y + Y = Y$$

(with the understanding that $\sum_{i=0}^{k-2} \alpha_i f^{i+1}(x)$ equals the origin if $k = 1$). Continuing this way inductively, we see that $f^m(x) \in Y$ for all $m \geq k$. But then $\text{orb}_f(x) \subseteq Y$, and hence, since the dimension of $Y$ is strictly less than that of $X$ (as $k - 1 < n$), it is impossible that $\text{orb}_f(x)$ be dense in $X$, contradicting $x$ being $f$-transitive.

**Proof of Proposition 5.2.** Take any positive integer $n$ and linear dynamical system $(\mathbb{R}^n, f)$, and suppose there is an $f$-transitive state $x$ in $\mathbb{R}^n$. Take any real number $\lambda$, and notice that there is a strictly increasing sequence $(m_k)$ of positive integers (that depend on $\lambda$) such that $f^{m_k}(x) \to \lambda x$ (because $\text{orb}_f(x)$ is dense in $\mathbb{R}^n$). Then, by Lemma 5.3, for any $n$-vector $y$, there are real numbers $\alpha_0, ..., \alpha_{n-1}$ such that $y = \sum_{i=0}^{n-1} \alpha_i f^i(x)$, and hence

$$f^{m_k}(y) = \sum_{i=0}^{n-1} \alpha_i f^i(f^{m_k}(x)) \to \sum_{i=0}^{n-1} \alpha_i f^i(\lambda x) = \lambda \sum_{i=0}^{n-1} \alpha_i f^i(x) = \lambda y.$$
In view of the arbitrariness of \( y \) here, we may then conclude that \( f^{m_k} \to \lambda \text{id}_{\mathbb{R}^n} \).

Applying what we have found in the previous paragraph (for \( \lambda = 0 \) and \( \lambda = 1 \)), we see that there are strictly increasing sequences \( (m_k) \) and \( (n_k) \) such that \( f^{m_k} \to 0 \) and \( f^{n_k} \to \text{id}_{\mathbb{R}^n} \). (Here \( 0 \) stands for the constant self-map on \( \mathbb{R}^n \) that maps any \( n \)-vector to the origin of \( \mathbb{R}^n \).) Then, where \( A \in \mathbb{R}^{n \times n} \) is the matrix representation of \( f \) – that is, \( f(z) = Az \) for every \( z \in \mathbb{R}^n \) – we have \( A^{m_k} \to 0_n \) and \( A^{n_k} \to \text{I}_n \), where \( 0_n \) and \( \text{I}_n \) are the \( n \times n \) zero and identity matrices, respectively. This implies \( \det(A)^{m_k} = \det(A^{m_k}) \to 0 \) and \( \det(A)^{n_k} = \det(A^{n_k}) \to 1 \) (by continuity of the determinant function; recall Example 1.14 of Chapter 2). But for any real number \( a \), the sequence \( (a^m) \) has a subsequence that converges to 1 iff \( |a| = 1 \), and hence, in that case, it cannot have a subsequence that converges to 0.

**Hypercyclicity in Infinite-Dimensional Spaces**

The situation is remarkably different in the context of infinite-dimensional linear dynamical systems. We illustrate this next.

**Example 5.1.** Consider the linear dynamical system \((\ell_p, \sigma_\lambda)\) where \( p \) and \( \lambda \) are real numbers with \( p \geq 1 \), and where \( \sigma_\lambda : \ell_p \to \ell_p \) is defined by

\[
\sigma_\lambda(x_1, x_2, \ldots) := (\lambda x_2, \lambda x_3, \ldots).
\]

(In other words, \( \sigma_\lambda \) is the \( \lambda \) multiple of the shift operator on \( \ell_p \).) If \( |\lambda| \leq 1 \), then

\[
\|\sigma_\lambda^m(x)\|_p = |\lambda|^m \|(x_{m+1}, x_{m+2}, \ldots)\|_p \leq \|x\|_p
\]

for any \( m \in \mathbb{N} \), and hence \( \text{orb}_{\sigma_\lambda}(x) \) is not dense, for any \( x \in \ell_p \). Thus: \( \sigma_\lambda \) is not hypercyclic whenever \( |\lambda| \leq 1 \).

Now assume \( |\lambda| > 1 \), and take any nonempty open subsets \( O \) and \( U \) of \( \ell_p \). As usual, let us denote that set of all real sequences only finitely many terms of which are nonzero by \( c_{00} \). It is readily verified that this set is dense in \( \ell_p \). Therefore, there are sequences of the form \( x := (x_1, \ldots, x_M, 0, 0, \ldots) \) and \( y := (y_1, \ldots, y_M, 0, 0, \ldots) \) in \( U \) and \( O \), respectively, where \( M \in \mathbb{N} \). Take any integer \( m \geq M \), and define \( z(m) \in c_{00} \) such that the \( k \)th term of \( z(m) \) equals \( x_k \) for any \( k = 1, \ldots, M \), it equals \( \lambda^{-m}y_{k-M} \) for any \( k = M + 1, \ldots, m + M \), and it equals 0 otherwise. Then, \( \sigma_\lambda^m(z(m)) = y \) for every \( m \geq M \). But since \( |\lambda| > 1 \), we have \( \|x - z(m)\|_p = |\lambda|^{-m} \|y\|_p \downarrow 0 \), that is, \( z(m) \to x \), as \( m \to \infty \).

It follows that \( z(m) \in U \) and \( \sigma_\lambda^m(z(m)) \in O \), that is, \( \sigma_\lambda^m(O) \cap U \neq \emptyset \), for large \( m \). Thus: \( (\ell_p, \sigma_\lambda) \) is topologically transitive. As we have noted in Proposition 5.1, this is the same thing as saying that \( \sigma_\lambda \) is hypercyclic.

49
Note. In the case of $(\ell^2, \sigma_2)$, the content of Example 5.1 was first established by Stefan Rolewicz in 1969 (by a different method). It is thus customary to refer to any self-map of the form $\sigma_\lambda$ on $\ell_p$ as a Rolewicz operator.

Example 5.1 is actually not pathological at all. Indeed, in 1997, Shamim Ansari has shown that for every infinite-dimensional separable Banach Space $X$, there is a continuous linear self-map on $X$ which is hypercyclic.\(^6\) As $\mathbb{R}^n$ can be replaced with any finite-dimensional normed linear space in Proposition 5.2 – we will prove this later in Chapter 12 – this result in fact yields an unexpected characterization of all finite-dimensional Banach spaces:

**Theorem 5.4.** (Ansari) Let $X$ be a separable Banach space. Then, there is an $f : X \to X$ such that $(X, f)$ is a topologically transitive linear dynamical system if, and only if, $X$ is infinite-dimensional.

**Remark.** Theorem 5.4 tells us that for any infinite-dimensional separable Banach space $X$, there is a continuous linear self-map $f$ on $X$ such that $\text{orb}_f(x)$ is dense in $X$ for some $x \in X$. It is even possible for some of such spaces that we can find a continuous linear self-map $f$ on $X$ such that $\text{orb}_f(x)$ is dense in $X$ for all $x \in X$. For instance, as proved by Charles Read in 1988 (by means of very elaborate computations), $\ell_1$ is such a space. In particular, there is a continuous linear map $f : \ell_1 \to \ell_1$ such that the only closed $f$-invariant subset of $\ell_1$ is $\ell_1$!

**Chaotic Linear Dynamical Systems**

It seems intuitive that a linear dynamical system cannot exhibit chaotic behavior, but it turns out that this is, again, a finite-dimensional intuition. In the context of infinite-dimensional state spaces, we may well encounter such behavior for continuous and linear transition functions. In fact, we can use Example 5.1 to demonstrate this possibility.

**Example 5.2.** Consider the linear dynamical system $(\ell_p, \sigma_\lambda)$ we looked at in Example 5.1, where $p$ and $\lambda$ are real numbers with $p \geq 1$ and $|\lambda| > 1$. Take any element $x$ of $c_{00}$, say, $x = (x_1, \ldots, x_M, 0, 0, \ldots)$, and for any $m > M$, define $x(m)$ to be the $m$-vector $(x_1, \ldots, x_M, 0, \ldots, 0)$. Now, for any $m > M$, put

$$y(m) := (x(m), \lambda^{-m}x(m), \lambda^{-2m}x(m), \ldots)$$

---

\(^6\)Ansari’s proof of this result is beyond the scope of the present text. For a more approachable, but still quite hard, proof, you might want to look at Bernal-Gonzáles (1999).
in the sense of vector concatenation. Clearly, $y(m) \in \text{Per}(\ell_p, \sigma_\lambda)$ for any $m > M$. Moreover,

$$
\|x - y(m)\|_p \leq \sum_{i=1}^{\infty} |\lambda|^{-im} \|x\|_p = \left( \frac{|\lambda|^{-m}}{1 - |\lambda|^{-m}} \right) \|x\|_p
$$

for any $m > M$, and hence $\|x - y(m)\|_p \to 0$. Conclusion: For every $x \in c_{00}$, there is a sequence in $\text{Per}(\ell_p, \sigma_\lambda)$ that converges to $x$ in $\ell_p$. Since $c_{00}$ is dense in $\ell_p$, it follows that $\text{Per}(\ell_p, \sigma_\lambda)$ is dense in $\ell_p$. Since we have seen in Example 5.1 that $(\ell_p, \sigma_\lambda)$ is topologically transitive, combining this observation with Theorem 2.6 we may conclude that $(\ell_p, \sigma_\lambda)$ is chaotic in the sense of Devaney.

**Linear vs. Nonlinear Dynamical Systems**

Despite the findings we have reported above, it is still quite natural to think that the dynamics of nonlinear systems are bound to be more complicated than linear systems. This is certainly true for linear systems with finite-dimensional states spaces. Nevertheless, and very surprisingly, the behavior of any dynamical system with a compact (metric) state space is captured fully by a subsystem of a (fixed!) linear dynamical system. This fact, due to Nathan Feldman, shows how useful linear dynamical systems are for the general theory of dynamical systems. We conclude the present chapter by proving this result.

**Theorem 5.5.** (Feldman) There exists a linear dynamical system $(X, f)$ with the following property: For any dynamical system $(Y, g)$ such that $Y$ is a compact metric space, there is an $f$-invariant subset $Z$ of $X$ with $(Z, f|_Z) \cong (Y, g)$.

**Proof.** Let $X$ stand for the set of all sequences in $\ell_2$ whose norms are square-summable, that is,

$$
X := \left\{ x : \mathbb{N} \to \ell^2 : \sum_{i=1}^{\infty} \|x(i)\|_2^2 < \infty \right\}.
$$

Define $\|\cdot\| : X \to \mathbb{R}$ by

$$
\|x\| := \left( \sum_{i=1}^{\infty} \|x(i)\|_2^2 \right)^{1/2}.
$$

We leave it as an exercise to prove that $\|\cdot\|$ is a norm on $X$ with respect to which $X$ is a separable Banach space. Next, we define $f : X \to X$ by
\( f(x)(i) := 2x(i + 1) \). As it is readily checked that \( f \) is continuous and linear, we conclude that \((X, f)\) is a linear dynamical system.

Now take any dynamical system \((Y, g)\) such that \(Y\) is a compact metric space (whose metric is denoted by \(d\)). Without loss of generality, we assume that the metric \(d\) of \(Y\) is bounded by 1 (otherwise we would replace it with the metric \(\min\{1, d\}\)). Since \(Y\) is a compact metric space, it is separable, so there is a countable dense subset of \(Y\) which we enumerate as \(\{y_1, y_2, \ldots\}\).

For any \(m \in \mathbb{Z}_+\), define \(\varphi_m : Y \to \ell_2\) by

\[
\varphi_m(y) := \left( \frac{1}{2}d(y_1, g^m(y)), \frac{1}{4}d(y_2, g^m(y)), \frac{1}{8}d(y_3, g^m(y)), \ldots \right),
\]

and \(\varphi : Y \to X\) by

\[
\varphi(y) := \left( \varphi_0(y), \frac{1}{2}\varphi_1(y), \frac{1}{4}\varphi_2(y), \frac{1}{8}\varphi_3(y), \ldots \right).
\]

We claim that \(\varphi\) is an embedding. Indeed, if \(\varphi(y) = \varphi(z)\) for some \(y, z \in Y\), then \(\varphi_0(y) = \varphi_0(z)\), and hence \(d(y_i, y) = d(y_i, z)\) for every \(i \in \mathbb{N}\). This implies \(y = z\) (because, since \(\{y_1, y_2, \ldots\}\) is dense in \(Y\), there is a strictly increasing sequence \((m_k)\) of positive integers with \(d(y_{m_k}, y) \to 0\); but then \((y_{m_k})\) converges to both \(y\) and \(z\)). Thus: \(\varphi\) is injective. In addition, for any \(y, z \in Y\) and any positive integer \(M\),

\[
\|\varphi(y) - \varphi(z)\|^2 = \sum_{i=0}^{\infty} 2^{-i} \|\varphi_i(y) - \varphi_i(z)\|^2
\]

\[
= \sum_{i=0}^{\infty} \sum_{k=1}^{\infty} 2^{-i-k} |d(y_k, g^i(y)) - d(y_k, g^i(z))|^2
\]

\[
\leq \sum_{i=0}^{M} \sum_{k=1}^{M} 2^{-i-k} |d(y_k, g^i(y)) - d(y_k, g^i(z))|^2 + \sum_{(i, k) \in A_M} 2^{-i-k}
\]

where \(A_M := \{(i, k) \in \mathbb{Z}_+ \times \mathbb{N} : i > M \text{ or } k > M\}\). Since \(\sum_{(i, k) \in A_M} 2^{-i-k} \uparrow 0\) as \(M \downarrow \infty\), therefore, we can make \(\|\varphi(y) - \varphi(z)\|^2\) arbitrarily small by choosing \(y\) and \(z\) close enough and \(M\) large enough. Thus: \(\varphi\) is continuous. We now apply the Homeomorphism Theorem to conclude that \(\varphi\) is an embedding.

We now put \(Z := \varphi(Y)\), and note that \(\varphi\) is a homeomorphism from \(Y\) onto \(Z\). Now, for any \(y \in Y\),

\[
\varphi_m(g(y)) = \left( \frac{1}{2}d(y_1, g^{m+1}(y)), \frac{1}{4}d(y_2, g^{m+1}(y)), \ldots \right) = \varphi_{m+1}(y)
\]

52
for every $m \in Z_+$, so

$$
\varphi(g(y)) = (\varphi_1(y), \frac{1}{2}\varphi_2(y), \frac{1}{3}\varphi_3(y), \ldots) = f(\varphi(y)),
$$

that is, $\varphi \circ g = f \circ \varphi$. Moreover, this implies that

$$
f(Z) = f(\varphi(Y)) = \varphi(g(Y)) \subseteq \varphi(Y) = Z,
$$

that is, $Z$ is an $f$-invariant subset of $X$. We have proved that $\varphi$ is a topological conjugacy between the dynamical systems $(Y, g)$ and $(Z, f|_Z)$.

The theory of linear dynamical systems is a deep and beautiful branch of operator theory, but going deeper into it requires a working knowledge of functional and complex analysis. We can do no better here than referring the interested reader to the excellent expositions of Bayart and Matheron (2009) and Grosse-Erdmann and Peris Manguillot (2011) for further study.

**Exercises**

**5.1.** Let $(X, f)$ be a linear dynamical system. Show that if $f$ is hypercyclic, then it exhibits sensitive dependence on initial conditions.

**5.2.** Let $(X, f)$ be a linear dynamical system. Prove that $f$ exhibits sensitive dependence on initial conditions if $\text{orb}_f(x)$ is an unbounded set for some $x \in X$. (Thus, the transition function of a finite-dimensional linear dynamical system can exhibit sensitive dependence on initial conditions (while it cannot be hypercyclic).)

**5.3.** Let $(w_m)$ be a sequence in $\mathbb{R}\setminus\{0\}$, and define the self-map $f$ on $\mathbb{R}^\infty$ by $f(x_1, x_2, \ldots) := (w_2x_2, w_3x_3, \ldots)$. Prove that $(\mathbb{R}^\infty, f)$ is a topologically transitive dynamical system.

**5.4.** (Kitai’s Criterion) Let $(X, f)$ be a linear dynamical system. Suppose that there is a dense subset $Z$ of $X$ and a sequence $(g_m)$ of self-maps on $X$ such that (i) $f^m(z) \to 0$ for every $z \in Z$; (ii) $g_m(z) \to 0$ for every $z \in Z$; and (iii) $f^m(g_m(z)) \to z$ for every $z \in Z$. Prove that $f$ is hypercyclic.

**5.5.** Prove that $(\ell_p, \sigma_\lambda)$ is topologically transitive for any real numbers $p$ and $\lambda$ with $p \geq 1$ and $|\lambda| > 1$ by using Kitai’s Criterion.

**5.6.** (Hypercyclicity Criterion) Let $(X, f)$ be a linear dynamical system. Suppose that there are dense subsets $Z$ and $W$ of $X$, a strictly increasing sequence $(m_k)$ in $\mathbb{N}$, and a sequence $(g_{m_k})$ of maps from $X$ into $W$ such that (i) $f^{m_k}(z) \to 0$ for every $z \in Z$; (ii) $g_{m_k}(w) \to 0$ for every $w \in W$; and (iii) $f^{m_k}(g_{m_k}(w)) \to w$ for every $w \in W$. Prove that $f$ is hypercyclic.
5.7. (Grosse-Erdmann and Peris) Let \((X, f)\) be a linear dynamical system, and for any nonempty \(O, U \in \mathcal{O}_X\), put \(N(O, U) := \{ k \in \mathbb{N} : f^k(O) \cap U \neq \emptyset \} \). (Recall Exercise 2.7.) Prove that \((X, f)\) is topologically mixing if and only if for any \(O \in \mathcal{O}_X\) and \(U \in \mathcal{O}_X(0)\), both \(N(O, U)\) and \(N(U, O)\) have finite complements in \(\mathbb{N}\).

5.8. Consider the linear dynamical system \((\ell_1, f)\) where \(f(x_1, x_2, \ldots) := (2x_2, \frac{3}{2}x_3, \frac{3}{4}x_4, \ldots)\). Prove that this system is topologically mixing but it is not chaotic in the sense of Devaney.

5.9. Let \((X, f)\) be a topologically transitive linear dynamical system. Show that every state in \(X\) is the sum of two \(f\)-transitive states in \(X\).

5.10. Let \((X, f)\) be a topologically transitive linear dynamical system. Prove that the set of all \(f\)-transitive states in \(X\) is connected. (We will solve this exercise in Chapter 12.)

5.11. Let \((X, f)\) be a topologically transitive linear dynamical system. Prove that the set of all \(f\)-transitive states in \(X\) is homeomorphic to \(X\).

Hints for Selected Exercises

Exercise 2.5. Replace the roles of \(O\) and \(U\) in the definition of topological transitivity.

Exercise 2.7. If there is an isolated point in \(X\), Exercise 2.6 settles the matter.

Exercise 2.10. \(O \cup f(O) \cup \cdots\) is \(f\)-invariant for any \(O \in \mathcal{O}_X\).

Exercise 2.11. Suppose otherwise, and derive a contradiction to what we found in Exercise 2.10.

Exercise 2.12. Check that \(\|F \circ f\|_\infty = |\lambda| \|F\|_\infty\), use the surjectivity of \(F\), and then invoke Exercise 2.11.

Exercise 2.16.c. Use Exercise 2.7.

Exercise 2.19.b. In this case, the torus \(X \times X\) splits into a collection of \(f_{a,0}\)-invariant circles (induced by the condition \(x_2 = 0\)) and the orbit of any one point in \(X \times X\) is dense in one of these circles.

Exercise 2.19.c. For the “only if,” part of the claim, suppose \(a \alpha + b \beta\) is an integer for some \((a, b) \neq (0, 0)\), consider the real map \(\varphi\) on \(X \times X\) defined by \(\varphi(x_1, x_2) := \sin 2\pi(ax_1 + bx_2)\), and use Exercise 2.4.

Exercise 2.20. By Lemma 2.12, \(f\) is surjective. Suppose there is an \(S \in \mathcal{C}_X \setminus \{X\}\) with \(f(S) = X\), and show that \(\bigcap_{i=0}^\infty f^{-i}(S)\) is closed and \(f\)-invariant, which means \((X, f)\) is not minimal.

Exercise 2.21. You may want to use Exercise 2.20.
Exercise 4.3. We know from the proof of Hirsch’s Theorem that \( \text{orb}_f(z) \) is a periodic orbit, and \( \text{orb}_f(z) \) attracts a nonempty open set. Show that if the period of \( z \) under \( f \) exceeds 1, there is a least upper bound for \( \text{orb}_f(z) \) in \( B_f(S) \), say, \( x \), and deduce that \( f(x) \approx x \). Then use the maximality of \( z \) to get \( x = z \). Now invoke Lemma 4.6 to get \( f(x) = x \).

Exercise 5.1. For any \( \varepsilon, \delta > 0 \), apply topological transitivity with respect to the sets \( O := \{ z \in X : \| z \| > \delta \} \) and \( U := \{ z \in X : \| z \| < \varepsilon \} \) to find a \( k \in \mathbb{Z}_+ \) and \( z \in f^k(O) \cap U \), and then compare any \( x \) in \( X \) with \( y := x + z \). Or, deduce this result from the Exercise 5.2.

Exercise 5.4. Use Proposition 5.1.

Exercise 5.5. Take \( Z \) as the set \( c_{00} \) and \( g_m := \lambda^{-m} g^m \), where \( g : \ell_p \to \ell_p \) is defined by \( g(x_1, x_2, ...) := (0, x_1, x_2, ...) \).

Exercise 5.9. Let \( S \) be the set of all \( f \)-transitive states in \( X \) and note that \( S \) is a dense \( G_\delta \)-set in \( X \). But then, for any \( x \in X \), \( x - S \) is also such a set, and hence, by the Baire Category Theorem, \( S \) and \( x - S \) intersect.

55