Appendix 1
Mathematical Analysis on \( \mathbb{R} \)

This appendix collects the main definitions and results that we need in the text in terms of classical mathematical analysis. Some of these will be generalized, and complemented in the next two appendices, but the present treatment is entirely elementary. Put specifically, we talk briefly about real sequences and infinite series, but pay due attention to the classical Bolzano-Weierstrass Theorem, subsequential limits, rearrangement of infinite series and the equality of iterated limits. We then move to the theory of real functions, review the notions of continuity and differentiability, and prove some classical results of calculus, such as the Mean Value Theorem and Taylor’s Theorem. The present chapter concludes with a primer on concave functions on the real line. The only prerequisites for this appendix are the basic terminology and concepts reviewed in Chapter A.

1 (Extended) Real Sequences

1.1 Convergence in \( \mathbb{R} \)

Convergent Sequences

Let \( (a_m) \) be a sequence of extended real numbers, that is, a sequence in \( \mathbb{R} \). (If each \( a_m \) is real, we refer to this sequence as a real sequence – the set of all real sequences is thus \( \mathbb{R}^\infty \).) For any real number \( a \), we say that \( (a_m) \) converges to \( a \) if for every \( \varepsilon > 0 \), there is an integer \( M \) such that \( |a_m - a| < \varepsilon \) for each \( m \geq M \). On the other hand, if for every \( b \in \mathbb{R} \), there is an integer \( M \) with \( a_m \geq b \) for each \( m \geq M \), we say that \( (a_m) \) converges to \( +\infty \), and if \( (-a_m) \) converges to \( -\infty \), we say that \( (a_m) \) converges to \( -\infty \). For any extended real number \( a \), we express the statement “\( (a_m) \) converges to \( a \)” by writing \( a_m \to a \), or \( \lim_{m \to \infty} a_m = a \), or \( \lim a_m = a \), in which case we refer to \( a \) as the limit of \( (a_m) \). It follows readily from the definitions that a sequence of extended real numbers may converge to at most one limit.

In words, when \( a \) is real, \( a_m \to a \) means that no matter how small \( \varepsilon > 0 \) is, all but finitely many terms of the sequence \( (a_m) \) are contained in the open interval \( (a - \varepsilon, a + \varepsilon) \). The idea is simply that the tail of a convergent real sequence approximates the limit of the sequence to any desired degree of accuracy. Finitely many terms of the sequence may be quite apart from its limit point, but eventually all of its terms accumulate around this limit. Similarly, \( a_m \to \infty \) means that, no matter how large a real number \( b \) we choose, all but finitely many terms of \( (a_m) \) are larger than \( b \).

Denseness of \( \mathbb{Q} \) in \( \mathbb{R} \)

A very important fact about \( \mathbb{R} \) is that every real number can be approximated by rational numbers to any desired degree of accuracy. Unsurprisingly, this is a consequence of the completeness of \( \mathbb{R} \).

**Lemma 1.1.** For any real numbers \( a \) and \( b \) with \( a < b \), there is a rational number \( r \) with \( a < r < b \).

**Proof.** Take any real numbers \( a \) and \( b \) with \( c := b - a > 0 \). To derive a contradiction, suppose that \( \{c, 2c, \ldots\} \) is bounded from above. Then, by the Completeness Axiom, \( s := \sup\{c, 2c, \ldots\} \) is a real number. But since \( c > 0 \), the number \( s - c \) is not an upper bound for \( \{c, 2c, \ldots\} \), that is, there exists a positive integer \( m^* \) such that \( s < (m^* + 1)c \). But this is impossible due to the definition of \( s \). We thus conclude that \( \{c, 2c, \ldots\} \) is not bounded from above. In particular, \( mc > 1 \), that is \( mb > ma + 1 \),
for some positive integer $m$. Now define $n := \min\{k \in \mathbb{Z} : k > ma\}$. Then, $ma < n \leq 1 + ma < mb$, and letting $r := n/m$ completes the proof.

**Proposition 1.2.** For any real number $a$, there is a sequence $(r_m)$ in $\mathbb{Q}$ such that $r_m \to a$.

**Proof.** Take any real number $a$, and use Lemma 1.1 to pick a rational number $r_m$ in the interval $(a, a + 1/m)$ for every positive integer $m$. Then, for any $\varepsilon > 0$, we have $|r_m - a| < 1/m < \varepsilon$ whenever $m \geq 1/\varepsilon$.

**Bounded Sequences**

We say that a sequence $(a_m)$ in $\mathbb{R}$ is **bounded from above** if there is a real number $K$ such that $a_i \leq K$ for each $i \in \mathbb{N}$. (In view of the completeness of the reals, this is the same thing as saying that $\sup\{a_1, a_2, \ldots\} < \infty$.) In turn, $(a_m)$ is said to be **bounded from below** if $(-a_m)$ is bounded from above, and it is called **bounded** if it is bounded from both above and below, that is, if $\sup\{|a_1|, |a_2|, \ldots\} < \infty$. Obviously, a bounded real sequence does not have to be convergent. (Example: $(−1, 1, −1, 1, \ldots)$.) But every real sequence that converges to a real number is sure to be bounded. Indeed, if all but finitely – say, $M$ – many terms of such a sequence is at most some $\varepsilon > 0$ away from some real number $a$, then this sequence is bounded either by $|a| + \varepsilon$ or by the largest of the absolute values of the first $M$ terms. Thus:

**Proposition 1.3.** Every convergent real sequence is bounded, but not conversely.

**Exercise 1.1.** (a) Use mathematical induction to prove Bernoulli’s Inequality: $(1 + t)^m \geq 1 + mt$ for any $(t, m) \in [-1, \infty) \times \mathbb{N}$.
(b) Use the Bernoulli’s Inequality to show that $(t^m)$ converges to $\infty$ if $t > 1$, and to $−\infty$ if $−1 > t$. In turn, use this finding to prove that $t^m \to 0$ for any real number $t$ with $|t| < 1$.
(c) Prove that $mt^m \to 0$ for any real number $t$ with $|t| < 1$.
(d) Prove that $\frac{a^m}{m!} \to 0$ for any real number $a > 0$.

**Exercise 1.2.** Let $(a_m)$ and $(b_m)$ be two real sequences such that $a_m \to a$ and $b_m \to b$ for some real numbers $a$ and $b$. Prove:
(a) $|a_m| \to |a|$; (b) $a_m + b_m \to a + b$;
(c) $a_m b_m \to ab$; (d) $1/a_m \to 1/a$, provided that $a, a_m \neq 0$ for each $m$.

**Exercise 1.3.** Let $(a_m)$, $(b_m)$ and $(c_m)$ be real sequences such that $a_m \leq b_m \leq c_m$ for each $m$. Show that if $a_m \to a$ and $c_m \to c$, then $a \leq c$. Moreover, if $a = c$, then $(b_m)$ converges to $a$ as well.

**Corollary.** For any extended real numbers $a$ and $b$ with $a \leq b$, and any convergent sequence $(a_m)$ in $[a, b]^{\infty}$, we have $a \leq \lim a_m \leq b$.

**Monotonic Sequences**

A sequence $(a_m)$ in $\mathbb{R}$ is said to be **(strictly) increasing** if $a_m \leq a_{m+1}$ $(a_m < a_{m+1})$ for each $m \in \mathbb{N}$. It is said to be **(strictly) decreasing** if $(-a_m)$ is (strictly) increasing. If it is either increasing or decreasing, $(a_m)$ is said to be **monotonic**. If $(a_m)$ is increasing and converges to $a \in \mathbb{R}$, then we write $a_m \uparrow a$, and if it is decreasing and converges to $a \in \mathbb{R}$, we write $a_m \downarrow a$.

The following observations attest to the importance of monotonic sequences.

**Proposition 1.4.** Every increasing (decreasing) real sequence in $\mathbb{R}$ that is bounded from above (below) converges to a real number. Consequently, every monotonic sequence in $\mathbb{R}$ converges in $\mathbb{R}$.
Proof. Let \((a_m)\) be an increasing sequence in \(\mathbb{R}\). If \((a_m)\) is not bounded from above, then for any \(b \in \mathbb{R}\) there is an integer \(M\) with \(a_M \geq b\), and as \((a_m)\) is increasing, this implies \(a_m \geq b\) for each \(m \geq M\). Thus, in this case \(a_m \uparrow \infty\). Assume now that \((a_m)\) is bounded from above, and let \(S := \{a_1, a_2, \ldots\}\). By the Completeness Axiom, \(a := \sup S\) is a real number. We wish to show that \(a_m \downarrow a\). To this end, pick an arbitrary \(\varepsilon > 0\). Since \(a\) is the least upper bound for \(S\), \(a - \varepsilon\) cannot be an upper bound of \(S\), so \(a_M > a - \varepsilon\) for some \(M \in \mathbb{N}\). Since \((a_m)\) is increasing, we must then have \(a \geq a_M \geq a - \varepsilon\), so \(|a_M - a| < \varepsilon\), for each \(m \geq M\). Thus: \(a_m \downarrow a\). The case of decreasing sequences is established by applying what we have found to \((-a_m)\).

The Bolzano-Weierstrass Theorem

Obviously, not every real sequence is monotonic. But still:

Proposition 1.5. Every real sequence has a monotonic subsequence.

Proof. Take any real sequence \((a_m)\), and define \(S_m := \{a_m, a_{m+1}, \ldots\}\) for each integer \(m \in \mathbb{N}\). If there is no maximum element in \(S_1\), then it is easy to see that \((a_m)\) has a monotonic subsequence. (Let \(a_{m_1} := a_1\), let \(a_{m_2}\) be the first term in the sequence \((a_2, a_3, \ldots)\) greater than \(a_{m_1}\), let \(a_{m_3}\) be the first term in the sequence \((a_{m_2+1}, a_{m_2+2}, \ldots)\) greater than \(a_{m_2}\), and so on.) By the same logic, if, for any positive integer \(m\), there is no maximum element in \(S_m\), we are done. Assume then \(\max S_m\) exists for each \(m \in \mathbb{N}\). Now we define the subsequence \((a_{m_k})\) recursively as follows: \(a_{m_1} := \max S_1\), \(a_{m_2} := \max S_{m_1+1}\), \(a_{m_3} := \max S_{m_2+1}\), and so on. Clearly, \((a_{m_k})\) is decreasing.

Putting Propositions 1.4 and 1.5 together yields the following fundamental result.

The Bolzano-Weierstrass Theorem. Every bounded real sequence has a convergent subsequence.

Exercise 1.4. Show that every unbounded real sequence has a subsequence that converges to either \(\infty\) or \(-\infty\).

Exercise 1.5.\(^{1}\) Let \(S\) be a nonempty subset of \(\mathbb{R}\). Show that there is an increasing sequence \((a_m) \in S^\infty\) such that \(a_m \uparrow \sup S\), and a decreasing sequence \((b_m) \in S^\infty\) such that \(b_m \downarrow \inf S\).

Exercise 1.6. For any real number \(a\) and \((a_m) \in \mathbb{R}^\infty\), show that \(a_m \to a\) iff every subsequence of \((a_m)\) has itself a subsequence that converges to \(a\).

1.2 Subsequential Limits

The long run behavior of a sequence is often studied by looking at those points to which at least one subsequence of that sequence converges. Given any \((a_m)\) in \(\overline{\mathbb{R}}\), we say that an extended real number \(a\) is a subsequential limit of \((a_m)\) if there exists a subsequence \((a_{m_k})\) of \((a_m)\) with \(a_{m_k} \to a\). (Example. \(-1, 1\) and \(\infty\) are the only subsequential limits of the sequence \((a_m)\) where \(a_m = -1\) for each odd \(m\) not divisible by 3, \(a_m = 1\) for each even \(m\), and \(a_m = m\) for each odd \(m\) divisible by 3.)

If \(a\) is a real number which is a subsequential limit of \((a_m)\), we understand that \((a_m)\) visits the interval \((a - \varepsilon, a + \varepsilon)\) infinitely often, that is, \(a_m\) is in this interval for infinitely many \(m\), no matter how small \(\varepsilon > 0\) is. It is in this sense that subsequential limits give us asymptotic information about the long run behavior of a real sequence. Of particular interest in this regard are the largest and smallest subsequential limits. These are called the limit superior (abbreviated as \(\limsup\)) and limit inferior (abbreviated as \(\liminf\)) of a real sequence. Formally, for any \((a_m)\) in \(\overline{\mathbb{R}}\), we define

\[
\limsup a_m := \sup \{ \lim a_{m_k} : (a_{m_k}) \text{ is a convergent subsequence of } (a_m) \}
\]

and

\[
\liminf a_m := \inf \{ \lim a_{m_k} : (a_{m_k}) \text{ is a convergent subsequence of } (a_m) \}.
\]
When doing otherwise may cause ambiguity, however, we will state explicitly the index that is led to infinity, and write $\limsup_{m \to \infty}$ instead of $\limsup$, and similarly for $\liminf$.

As $\bar{\mathbb{R}}$ is a complete lattice, the $\limsup$ and $\liminf$ of a sequence $(a_m)$ of extended real numbers are sure to exist. Moreover, as $\sup S = -\inf(-S)$ and $\inf S = -\sup(-S)$ for any subset $S$ of $\bar{\mathbb{R}}$, there is a useful connection between these extended real numbers:

$$\limsup a_m = -\liminf(-a_m) \quad \text{and} \quad \liminf a_m = -\limsup(-a_m).$$  \hspace{1cm} (1)

This relation often allows one to work out a particular result only for $\limsup$ of a sequence, and obtain the corresponding result for $\liminf$ without repeating the argument needlessly.

**Proposition 1.6.** Let $(a_m)$ be a sequence in $\bar{\mathbb{R}}$. Then,

$$\limsup a_m \geq \liminf a_m.$$  

Moreover, this inequality holds as an equality if and only if $(a_m)$ converges.

**Proof.** The first assertion is obvious. Moreover, as every subsequence of a convergent sequence converges to the limit of the sequence, this inequality holds as an equality for any convergent $(a_m)$. Conversely, suppose $a$ is an extended real number that equals both $\limsup a_m$ and $\liminf a_m$. Take any subsequence of $(a_m)$. By Proposition 1.5, this subsequence has a monotonic subsequence which, by Proposition 1.4, converges in $\bar{\mathbb{R}}$. Obviously, the limit of this latter subsequence must be $a$. We thus proved that every subsequence of $(a_m)$ has itself a subsequence that converges to $a$. This is the same thing as saying that $a_m \to a$ (Exercise 1.6).

The following result provides a characterization of the $\limsup$ and $\liminf$ of a sequence which is sometimes easier to work with.

**Proposition 1.7.** Let $(a_m)$ be a sequence in $\bar{\mathbb{R}}$. Then,

$$\limsup a_m = \inf\{\sup\{a_m, a_{m+1}, \ldots\} : m \in \mathbb{N}\}$$

and

$$\liminf a_m = \sup\{\inf\{a_m, a_{m+1}, \ldots\} : m \in \mathbb{N}\}.$$  

**Proof.** Let $A$ be the set of all subsequences of $(a_m)$ that converge in $\bar{\mathbb{R}}$, and put $s_m := \sup\{a_m, a_{m+1}, \ldots\}$ for each positive integer $m$. Note first that for any $(a_{m_k})$ in $A$, we have $s_{m_k} \geq a_{m_k}$ for each $k \geq 1$. But as $(s_m)$ is a decreasing sequence, it, and hence any of its subsequences, converge to $\inf s_m$. Consequently, for any $(a_{m_k})$ in $A$, we have $\inf s_m \geq \lim a_{m_k}$, and it follows that $\inf s_m \geq \limsup a_m$. To prove the reverse inequality, pick any $\varepsilon > 0$, and choose any $m_1 \in \mathbb{N}$ such that $a_{m_1} + \varepsilon > s_1$. (Such an $a_{m_1}$ exists by definition of $s_1$.) Next, choose any integer $m_2 > m_1$ such that $a_{m_2} + \varepsilon > s_{m_1+1}$. (Such an $a_{m_2}$ exists by definition of $s_{m_1+1}$.) Continuing this way inductively, we obtain a subsequence $(a_{m_k})$ of $(a_m)$ such that $s_{m_{k+1}} + \varepsilon \geq a_{m_k} + \varepsilon > s_{m_k+1}$ for each $k \geq 1$. As $s_m \downarrow \inf s_m$, therefore, we find that $(a_{m_k})$ is an element of $A$ with $a_{m_k} \to \inf s_m$. It then follows from the definition of $\limsup a_m$ that $\limsup a_m \geq \inf s_m$, and we are done with the first part of the proposition. On the other hand, the second part of this proposition is proved readily by using its first part and (1).

**Exercise 1.7.** Prove that, for any extended real number $a$ and $(a_m) \in \bar{\mathbb{R}}^\infty$, we have $a = \limsup a_m$ iff (i) for any $\varepsilon > 0$, there is an $M \in \mathbb{N}$ such that $a_m < a + \varepsilon$ for all $m \geq M$, and (ii) for any $\varepsilon > 0$ and $m \in \mathbb{N}$, there is an integer $k > m$ such that $a_k > a - \varepsilon$. (Also state and prove the analogous result for the $\liminf$ of $(a_m)$.) Conclude that $\limsup a_m < \infty$ implies that $(a_m)$ is bounded from above.
Exercise 1.8. Show that every real sequence \((a_m)\) has a monotonic subsequence \((a_{m_k})\) such that \(a_{m_k} \to \lim \sup a_m\).

Exercise 1.9. Prove: For any two bounded real sequences \((a_m)\) and \((b_m)\), we have
\[
\lim \inf a_m + \lim \inf b_m \leq \lim \inf (a_m + b_m).
\]

(Note. The inequality goes the other direction in the case of the limit superior.) Also give an example to show that this inequality may hold strictly.

### 1.3 Infinite Series

#### Convergence, Divergence, and Non-Existence of Infinite Series

By an infinite series, we mean a real sequence of the form \((a_1 + \cdots + a_m)\) for some \((a_m) \in \mathbb{R}^\infty\). When the limit of this sequence exists in \(\mathbb{R}\), we denote it as \(\sum_{i=1}^{\infty} a_i\), but, again, we write \(\sum a_i\) for \(\sum_{i=1}^{\infty} a_i\) within the text. That is,
\[
\sum_{i=1}^{\infty} a_i := \lim_{m \to \infty} \sum_{i=1}^{m} a_i,
\]
provided that the sequence \((a_1 + \cdots + a_m)\) converges in \(\mathbb{R}\). Similarly,
\[
\sum_{i=k}^{\infty} a_i := \sum_{i=k}^{\infty} a_{i+k-1}, \quad k = 1, 2, \ldots,
\]
provided that the right-hand side is well-defined. We say that the infinite series \((a_1 + \cdots + a_m)\) is convergent, if \(\sum_{i=1}^{\infty} a_i\) is a real number. In this case, with a standard abuse of terminology, we say that “the series \(\sum_{i=1}^{\infty} a_i\) is convergent.” If the infinite series \((|a_1| + \cdots + |a_m|)\) is convergent, we say that \((a_1 + \cdots + a_m)\) is absolutely convergent, or that “the series \(\sum_{i=1}^{\infty} a_i\) is absolutely convergent.” This is, of course, the same thing as saying that \(\sum_{i=1}^{\infty} |a_i| < \infty\).

**Warning.** If \((a_m)\) is a sequence of nonnegative real numbers, then \(\sum_{i=1}^{\infty} a_i\) is surely well-defined; it is a number in \([0, \infty]\). But, in general, \(\sum_{i=1}^{\infty} a_i\) may not be well-defined. (Example. The sequence \((\sum_{i=1}^{m} (-1)^{i})\) does not have a limit, so the notation \(\sum_{i=1}^{\infty} (-1)^{i}\) is meaningless.) Before dealing with an object like \(\sum_{i=1}^{\infty} a_i\) in practice, one should first make sure that it is well-defined.

**Remark 1.1.** For any real sequence \((a_m)\), the convergence of \(\sum_{i=1}^{\infty} a_i\) implies \(a_m \to 0\). For,
\[
\lim a_m = \lim \left( \sum_{i \in [m+1]} a_i - \sum_{i \in [m]} a_i \right) = \lim \sum_{i \in [m+1]} a_i - \lim \sum_{i \in [m]} a_i = 0
\]
where the second equality follows from Exercise 1.2. However, \(a_m \to 0\) does not imply that \(\sum_{i=1}^{\infty} a_i\) converges. For instance, we have \(\sum_{i=1}^{\infty} \frac{1}{i} = \infty\). To see this, let \((b_m) := \left(\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \cdots\right)\), and notice that \(\frac{1}{i} \geq b_i\) for each \(i\), and hence \(\sum_{i=1}^{\infty} \frac{1}{i} \geq \sum_{i=1}^{\infty} b_i = \frac{1}{2} + \frac{1}{2} + \cdots = \infty\). □

**Examples**

Given the nature of this course, we will not make arduous computations with infinite series. All we need in terms of specific infinite series is contained in the following two examples.
Example 1.1. For any real number $p$ with $-1 < p < 1$, we have

$$
\sum_{i=1}^{\infty} p^i = \frac{p}{1-p} \quad \text{and} \quad \sum_{i=1}^{\infty} ip^i = \frac{p}{(1-p)^2}.
$$

In particular, we have

$$
\sum_{i=1}^{\infty} 2^{-i} = 1 \quad \text{and} \quad \sum_{i=1}^{\infty} i2^{-i} = 2.
$$

Both formulas in (2) are proved by the same technique. First, note that $(1+p+\cdots+p^m)(1-p) = 1 - p^{m+1}$ holds for every positive integer $m$ and $p \in \mathbb{R}$, which is a fact that is readily established by mathematical induction. But, by Exercise 1.1, $(1 - p^{m+1})$ converges to 1 for any $p$ in $(-1,1)$, and hence the first formula in (2). Next, check, again by using mathematical induction, that $(p + 2p^2 + \cdots + mp^m)(1-p)$ equals $p + p^2 + \cdots + p^m - mp^{m+1}$ for every positive integer $m$ and $p \in \mathbb{R}$. But, for any $p$ in $(-1,1)$, we know from Exercise 1.1.(c) that $mp^{m+1} \to 0$, so letting $m \to \infty$ yields $(p + 2p^2 + \cdots + mp^m)(1-p) \to \sum_{i=1}^{\infty} p^i$. Invoking now the first formula in (2) yields the second formula that we are after.

Example 1.2. Let $k$ be any integer with $k \geq 2$. Then $\sum_{i=1}^{\infty} i^{-k}$ is a convergent series. To prove this, take any positive integer $n$ and choose an $m \in \mathbb{N}$ with $2^{m+1} > n$. Note that

$$
\sum_{i \in [n]} i^{-k} < 1 + (2^{-k} + 3^{-k}) + \cdots + ((2^m)^{-k}) + \cdots + ((2^{m+1} - 1)^{-k})
$$

$$
= 1 + 2^{-k} + 2^{-k} + \cdots + ((2^m)^{-k}) + \cdots + ((2^{m+1} - 1)^{-k})
$$

$$
= 1 + 2^{-k} + \cdots + (2^{1-k} m).
$$

Using Example 1.1 (with $p$ being $2^{1-k}$), we thus see that $\sum_{i \in [n]} i^{-k}$ is bounded above by $1/(1-2^{1-k})$. As $n$ is arbitrary here, we conclude that $\sum_{i=1}^{\infty} i^{-k}$ is convergent. (However, finding exactly what $\sum_{i=1}^{\infty} i^{-k}$ equals is not an easy matter.)

Exercise 1.10. For any convergent infinite series ($\sum_{i=1}^{\infty} a_i$), prove:

$$
\lim_{m \to \infty} \sum_{i=m}^{\infty} a_i = 0 \quad \text{and} \quad \left| \sum_{i=k}^{\infty} a_i \right| \leq \sum_{i=k}^{\infty} |a_i| \quad \text{for each} \quad k \in \mathbb{N}.
$$

Exercise 1.11. Let $(a_m)$ be a sequence in $\mathbb{R}_+$ such that $a_{m+1}/a_m \to a$ for some real number $a$. Show that $\sum_{i=1}^{\infty} a_i$ is convergent if $a < 1$, and $\sum_{i=1}^{\infty} a_i = \infty$ if $a > 1$. (The situation is indeterminate when $a = 1$.)

Summing Two Infinite Series

Let $(a_m)$ and $(b_m)$ be two real sequences. If $\sum_{i=1}^{\infty} a_i$ and $\sum_{i=1}^{\infty} b_i$ are convergent, then it readily follows from Exercise 1.2.(b) that $\sum_{i=1}^{\infty} (a_i + b_i)$ converges as well, and we have

$$
\sum_{i=1}^{\infty} (a_i + b_i) = \sum_{i=1}^{\infty} a_i + \sum_{i=1}^{\infty} b_i.
$$

It is also easily checked that the same formula holds if both $\sum_{i=1}^{\infty} a_i$ and $\sum_{i=1}^{\infty} b_i$ equal $\infty$, or both equal $-\infty$.  


Rearrangement Invariance of Nonnegative Series

An issue that arises frequently in practice concerns the rearrangement of an infinite series. The question is if one can sum the terms of a given real sequence in different orders and obtain the same result (as it would be the case for any real n-vector). The answer is no, in general, but a positive answer ensues in many interesting cases. In particular, there is no room for worry in the case of a series with nonnegative terms.

**Proposition 1.8.** Let \( \rho \) be a permutation on \( \mathbb{N} \), and \( (a_m) \) a real sequence with \( a_m \geq 0 \) for each \( m \). Then, \( \sum_{i=1}^{\infty} a_i = \sum_{i=1}^{\infty} a_{\rho(i)} \).

**Proof.** Let \( m \) be a positive integer. Since \( \rho \) is bijective, there exist integers \( K_m \) and \( L_m \) with \( K_m \geq L_m \geq m \) and \( [m] \subseteq \{\rho(1), ..., \rho(L_m)\} \subseteq [K_m] \). So, since \( a_i \geq 0 \) for each \( i \),

\[
\sum_{i \in [m]} a_i \leq \sum_{i \in [L_m]} a_{\rho(i)} \leq \sum_{i \in [K_m]} a_i.
\]

Letting \( m \to \infty \) yields the claim. \( \blacksquare \)

Rearrangement Invariance of Absolutely Convergent Series

The following result is often invoked when Proposition 1.8 does not apply because the series at hand may have terms that alternate in sign.

**Dirichlet’s Rearrangement Theorem.** Let \( \rho \) be a permutation on \( \mathbb{N} \), and \( (a_m) \) a real sequence with \( \sum_{i=1}^{\infty} |a_i| < \infty \). Then, \( \sum_{i=1}^{\infty} a_i = \sum_{i=1}^{\infty} a_{\rho(i)} \).

**Proof.** For any real number \( t \), we define \( t^+ := \max \{0, t\} \) and \( t^- := \max \{0, -t\} \). (Note. \( \min\{t^+, t^-\} \geq 0 \) and \( t = t^+ - t^- \).) Since \( \sum_{i=1}^{\infty} a_{\rho(i)}^+ \leq \sum_{i=1}^{\infty} |a_{\rho(i)}| \leq \sum_{i=1}^{\infty} a_i < \infty \) for every positive integer \( m \), it must be the case that \( \sum_{i=1}^{\infty} a_{\rho(i)}^+ \) is convergent (Proposition 1.4). Similarly, any one of the infinite series \( \sum_{i=1}^{\infty} a_i^+ \), \( \sum_{i=1}^{\infty} a_{\rho(i)}^+ \) and \( \sum_{i=1}^{\infty} a_{\rho(i)}^- \) converges. Therefore,

\[
\sum_{i=1}^{\infty} a_i = \sum_{i=1}^{\infty} (a_i^+ - a_i^-) = \sum_{i=1}^{\infty} a_i^+ - \sum_{i=1}^{\infty} a_i^-
\]

and

\[
\sum_{i=1}^{\infty} a_{\rho(i)} = \sum_{i=1}^{\infty} (a_{\rho(i)}^+ - a_{\rho(i)}^-) = \sum_{i=1}^{\infty} a_{\rho(i)}^+ - \sum_{i=1}^{\infty} a_{\rho(i)}^-
\]

by part (b) of Exercise 1.2. But, by Proposition 1.8, we have \( \sum_{i=1}^{\infty} a_i^+ = \sum_{i=1}^{\infty} a_{\rho(i)}^+ \) and \( \sum_{i=1}^{\infty} a_i^- = \sum_{i=1}^{\infty} a_{\rho(i)}^- \), and hence \( \sum_{i=1}^{\infty} a_i = \sum_{i=1}^{\infty} a_{\rho(i)} \). \( \blacksquare \)

Let \( f \) be a real map on a countably infinite set \( A \). As we have noted in Remark A.3.1, by the expression

\[
\sum_{a \in A} f(a),
\]

we mean the infinite series \( \sum_{i=1}^{\infty} f(a_i) \), where \( \{a_1, a_2, \ldots\} \) is an arbitrarily chosen enumeration of \( A \). Thus, this expression is meaningful only if \( \sum_{i=1}^{\infty} f(a_i) = \sum_{i=1}^{\infty} f(a_{\rho(i)}) \) for any permutation \( \rho \) on \( \mathbb{N} \). It thus follows from Proposition 1.8 and Dirichlet’s Rearrangement Theorem that the expression \( \sum_{a \in A} f(a) \) is meaningful, provided that either \( f \geq 0 \) or \( \sum_{i=1}^{\infty} f(a_i) \) is absolutely convergent. The following result is a consequence of this understanding.
Corollary 1.9. Let \( f \) and \( g \) be two real maps on a countably infinite set \( A \) such that \( \sum_{a \in A} |f(a)| < \infty \) and \( \sum_{a \in A} |g(a)| < \infty \). Then,

\[
\sum_{a \in A} (f(a) + g(a)) = \sum_{a \in A} f(a) + \sum_{a \in A} g(a).
\]

Proof. As we noted above, the sum of two convergent infinite series is the infinite series obtained by summing the corresponding terms of those series. Consequently, all we need to do here is to make sure that the left-hand side of our claim is well-defined. To this end, take any enumeration \( \{a_1, a_2, \ldots \} \) of \( A \), and notice that

\[
\sum_{i=1}^{\infty} |f(a_i) + g(a_i)| \leq \sum_{i=1}^{\infty} (|f(a_i)| + |g(a_i)|) = \sum_{i=1}^{\infty} |f(a_i)| + \sum_{i=1}^{\infty} |g(a_i)| < \infty.
\]

It thus follows from Dirichlet’s Rearrangement Theorem that \( \sum_{a \in A} (f(a) + g(a)) \) is meaningful, and we are done. \( \square \)

*Exercise 1.12. (Riemann’s Rearrangement Theorem) Let \( (a_m) \) be a real sequence such that \( \sum_{n=1}^{\infty} a_n \) converges to a real number and \( \sum_{n=1}^{\infty} |a_n| = \infty \). Prove: For every real number \( a \) there is a permutation \( \rho \) on \( \mathbb{N} \) such that \( \sum_{n=1}^{\infty} a_{\rho(n)} = a \).

1.4 Infinite Products

Let \( (a_m) \) be a real sequence. We define \( \prod_{i \in [m]} a_i := a_1 \cdots a_m \) for any positive integer \( m \). By an infinite product, we mean a real sequence of the form \( (a_1, a_1, a_2, \ldots) \) for some \( (a_m) \in \mathbb{R}^\infty \). When the limit of this sequence exists in \( \mathbb{R} \) we denote it as \( \prod_{i=1}^{\infty} a_i \), but again, we often write \( \prod_{i=1}^{\infty} a_i \) for \( \prod_{i=1}^{\infty} a_i \) within the text to simplify our notation. That is,

\[
\prod_{i=1}^{\infty} a_i := \lim_{m \to \infty} \prod_{i \in [m]} a_i,
\]

provided that the sequence \( (\prod_{i \in [m]} a_i) \) converges in \( \mathbb{R} \). We say that the infinite product \( \prod_{i \in [m]} a_i \), or abusing terminology, \( \prod_{i=1}^{\infty} a_i \), is convergent if \( \lim (\prod_{i \in [m]} a_i) \) is a real number. If \( (\prod_{i \in [m]} a_i) \) converges to either \( \infty \) or \( -\infty \), that is, \( \prod_{i=1}^{\infty} a_i \in \{-\infty, \infty\} \), we say that the infinite product, or \( \prod_{i=1}^{\infty} a_i \), is divergent.

1.5 Double Real Sequences

By a double real sequence, we mean a double sequence in \( \mathbb{R} \). As we have noted in Section A.1.8, it is customary to denote the collection of all double real sequences by \( \mathbb{R}^{\infty \times \infty} \) (instead of \( \mathbb{R}^{\mathbb{N} \times \mathbb{N}} \)). A generic member of this collection is denoted as \((a_{mn})\) with the understanding that the value of the sequence at \((m, n)\) is the real number \(a_{mn}\). For any real number \(a\), we say that such a double sequence \((a_{mn})\) converges to \(a\) if for each \(\varepsilon > 0\), there is an integer \(M\) such that \(|a_{mn} - a| < \varepsilon\) for all integers \(m, n \geq M\). This means that no matter how small \(\varepsilon > 0\) is, the terms of the sequence \((a_{mn})\) that correspond to sufficiently large \(m\) and \(n\) are contained in the open interval \((a - \varepsilon, a + \varepsilon)\).

If \((a_{mn})\) converges to a real number \(a\), we say that \((a_{mn})\) is convergent (or that it converges in \( \mathbb{R} \)), and refer to \(a\) as the limit of \((a_{mn})\). We describe this situation by writing \(a_{mn} \to a\) or \(\lim a_{mn} = a\). (It is plain that a double real sequence may have at most one limit.) On the other hand, if for every \(b \in \mathbb{R}\), there is an integer \(M\) with \(a_{mn} \geq b\) for each \(m, n \geq M\), we say that \((a_{mn})\) converges to \(\infty\), or that “the limit of \((a_{mn})\) is \(\infty\),” and write either \(a_{mn} \to \infty\) or \(\lim a_{mn} = \infty\).
We say that \((a_{mn})\) converges to \(\pm \infty\), and write \(a_{mn} \to \pm \infty\) or \(\lim a_{mn} = \pm \infty\), if \(-a_{mn} \to \infty\). If \((a_{mn})\) converges to a real number, or \(\infty\), or \(-\infty\), we say that it converges in \(\mathbb{R}\).

**Iterated Limits of Double Real Sequences**

Let \((a_{mn})\) be a double real sequence. Then, for any fixed \(m\), \((a_{m1}, a_{m2}, \ldots)\) is a real sequence, so we can talk about its limit, say, \(a(m)\). Assuming that \(a(m)\) exists in \(\mathbb{R}\) for each \(m\), then, we can talk about the limit of the sequence \((a(m))\) in \(\mathbb{R}\). This limit, if it exists, is denoted by

\[
\lim_{m \to \infty} \lim_{n \to \infty} a_{mn},
\]

while we define the extended real number \(\lim_{n \to \infty} \lim_{m \to \infty} a_{mn}\) analogously (when it exists). These numbers are called the iterated limits of \((a_{mn})\), and are in general distinct. (Example. Consider \((\frac{m}{m+n})\).) A natural question is if these numbers are the same at least when \((a_{mn})\) is convergent. The answer is again no. (Example. Consider \(((\pm 1)^m (\frac{1}{m} + \frac{1}{n}))\).) Conversely, we may ask if \((a_{mn})\) is convergent when these iterated limits are known to be the same. Unfortunately, the answer is again negative. (Example. Consider \((\frac{mn}{m^2+n^2})\).) Evidently, there is room for caution when dealing with iterated limits of a double real sequence. The following result is often useful in this regard.

**Proposition 1.10.** Let \((a_{mn})\) be a double real sequence that converges in \(\mathbb{R}\). If, for every positive integer \(m\), the real sequence \((a_{m1}, a_{m2}, \ldots)\) converges in \(\mathbb{R}\), then

\[
\lim_{m \to \infty} \lim_{n \to \infty} a_{mn} = \lim_{n \to \infty} \lim_{m \to \infty} a_{mn}.
\]

**Proof.** Take any \(\varepsilon > 0\), put \(a := \lim a_{mn}\), and choose an integer \(M\) such that \(|a_{mn} - a| < \frac{\varepsilon}{2}\) whenever \(m, n \geq M\). Now, for each \(m\), put \(a(m) := \lim_{n \to \infty} a_{mn}\) which is a real number by hypothesis. Then, for each \(m\), there is a positive integer \(n_m \geq M\) such that \(|a(m) - a_{mn_m}| < \frac{\varepsilon}{2}\), and hence, for any integer \(m \geq M\), we have \(|a(m) - a| \leq |a(m) - a_{mn_m}| + |a_{mn_m} - a| < \varepsilon\). We may thus conclude that \(a(m) \to a\), as was to be proved.

**On the Equality of Iterated Infinite Series**

A question that is frequently encountered when dealing with partial sums of the terms of a double real sequence is if we would get the same answer by computing the value of the iterated series in different orders. More precisely, given a double real sequence \((a_{mn})\), the problem is understanding whether we have

\[
\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij} = \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} a_{ij}.
\]

(3)

or not. A moment’s reflection shows that this is a particular form of “interchanging the iterated limits” problem we have mentioned above. After all, (3) is the same thing as saying that

\[
\lim_{m \to \infty} \lim_{n \to \infty} \sum_{i \in [m]} \sum_{j \in [n]} a_{ij} = \lim_{n \to \infty} \lim_{m \to \infty} \sum_{i \in [m]} \sum_{j \in [n]} a_{ij}.
\]

As such, it is not surprising that this equality is valid for nonnegative double real sequences. The following fact confirms this, and says a bit more. It turns out that, when \(a_{mn} \geq 0\) for each \(m\) and \(n\), not only are the iterated limits above the same, but they equal the extended real number we would obtain by summing \(a_{mn}\) in any order.

**The Discrete Version of Tonelli’s Theorem.** Let \((a_{mn})\) be a double real sequence with \(a_{mn} \geq 0\) for each \(m\) and \(n\). Then,

\[
\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij} = \sum_{(i,j) \in \mathbb{N}^2} a_{ij} = \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} a_{ij}.
\]
Proof. By symmetry, it is enough to establish the first equality alone. To this end, define $b_{mn} := \sum_{i \in [m]} \sum_{j \in [n]} a_{ij}$ for any positive integers $m$ and $n$, and let $b := \sum_{(i,j) \in \mathbb{N}^2} a_{ij}$. (Recall Remark A.3.1, and notice that $b$ is well-defined (in $[0, \infty]$) by Proposition 1.4.) Obviously, $b \geq b_{mn}$ for each $m$ and $n$, so $b \geq \lim_{m \to \infty} \lim_{n \to \infty} b_{mn}$. On the other hand, by Proposition 1.8,

$$b = a_{11} + (a_{12} + a_{21} + a_{22}) + (a_{13} + a_{31} + a_{23} + a_{32} + a_{33}) + \cdots = \sup \{b_{kk} : k \in \mathbb{N} \}.$$  

But for any $k$, we have $b_{mn} \geq b_{kk}$ for every $m, n \geq k$, and it follows that $b \leq \lim_{m \to \infty} \lim_{n \to \infty} b_{mn}$. □

Warning. The nonnegativity assumption cannot be omitted in the statement of Proposition 1.10. (Example. Consider the double real sequence $(a_{mn})$ where $a_{ij} := 1$ if $j > i$, $a_{ij} := 0$ if $i = j$, and $a_{ij} := -1$ if $j < i$.)

Corollary 1.11. For any sequence $(c_m)$ in $\mathbb{R}_+$, we have

$$\sum_{i=1}^\infty \sum_{j=i}^\infty c_j = \sum_{j=1}^\infty \sum_{i=j}^\infty j c_j.$$  

Proof. For any positive integers $m$ and $n$, put $a_{mn} := c_n$ if $n \geq m$, and $a_{mn} := 0$ otherwise. Now apply the Discrete Version of Tonelli’s Theorem. □

In view of Remark A.3.1, and because the product of two countable sets is countable (Corollary A.3.4), the Discrete Version of Tonelli’s Theorem may be restated in the following more abstract form.

Corollary 1.12. Let $A$ and $B$ be two countable sets, and $f : A \times B \to \mathbb{R}_+$ any function. Then,

$$\sum_{a \in A} \sum_{b \in B} f(a, b) = \sum_{(a,b) \in A \times B} f(a, b) = \sum_{b \in B} \sum_{a \in A} f(a, b).$$

2 Continuity and Differentiation on $\mathbb{R}$

2.1 Continuity of Real Maps on $\mathbb{R}$

Limits of a Real Map on an Interval

Let $I$ be a nondegenerate interval in $\mathbb{R}$ and $f \in \mathbb{R}^I$. For any $t$ in $I \cup \{\inf I\}$ which is distinct from $\max I$, we say that an extended real number $c$ is the right-limit of $f$ at $t$, if $f(t_m) \to c$ for every sequence $(t_m)$ in $I$ with $t_m \downarrow t$. (Notice that $f$ does not have to be defined at $t$.) If $t > -\infty$ here, we denote the right-limit of $f$ at $t$ by $f(t^+)$, and if $t = -\infty$, by $f(-\infty)$. For any $t$ in $I \cup \{\sup I\}$ which is distinct from $\min I$, the left-limit of $f$ at $t$, denoted by $f(t^-)$ if $t < \infty$, and by $f(\infty)$ if $t = \infty$, is defined analogously. Finally, if $t$ is a real number in $I \cup \{\inf I, \sup I\}$ which is distinct from $\max I$ and $\min I$, we define the limit of $f$ at $t$ as the extended real number $c$ such that $f(t_m) \to c$ for every sequence $(t_m)$ in $I$ with $t_m \to t$. When it exists, we denote this number $c$ by $\lim_{x \to t} f(s)$. In particular, when $I$ is an open interval and $t \in I$, we have $\lim_{x \to t} f(s) = c$ if $f(t^+) = c = f(t^-)$.

Continuity of a Real Map on an Interval

For any $t$ in $I \setminus \{\sup I\}$, we say that $f$ is right-continuous at $t$ if $f(t^+) = f(t)$, and similarly, for any $t$ in $I \setminus \{\inf I\}$, we say that $f$ is left-continuous at $t$ if $f(t^-) = f(t)$. Globally, we say that $f$
is right-continuous if it is right-continuous at every \( t \in I \backslash \{ \sup I \} \), and similarly, \( f \) is said to be left-continuous if it is left-continuous at every \( t \in I \backslash \{ \inf I \} \).

We say that \( f \) is continuous at \( t \in I \), if either (i) \( t = \sup I \) and \( f \) is left-continuous at \( t \), or (ii) \( t = \inf I \) and \( f \) is right-continuous at \( t \), or (iii) \( t \) equals neither \( \sup I \) nor \( \inf I \) and \( f \) is both right- and left-continuous at \( t \). (This is the same thing as saying that for every \( \varepsilon > 0 \), there is a \( \delta > 0 \) such that \( |f(s) - f(t)| < \varepsilon \) for every \( s \in I \) with \( |s - t| < \delta \).) Then, for any nonempty subset \( S \) of \( I \), we say that \( f \) is continuous on \( S \) if \( f \) is continuous at every \( t \in S \). Finally, if \( f \) is continuous on \( I \), we simply say that \( f \) is continuous. It is readily verified that continuity of \( f \) means that \( f(\lim t_m) = \lim f(t_m) \) for every sequence \( (t_m) \) in \( I \) that converges in \( I \).

**Notation.** The set of all continuous functions on an interval \( I \) is denoted by \( C(I) \), but if \( I \) equals an interval of the form \( [a, b] \), we write \( C[a, b] \) instead of \( C([a, b]) \).

**Example 2.1.** A self-map on \( \mathbb{R} \) may fail to be continuous at every point in its domain. For instance, \( 1_\mathbb{Q} \), the indicator function of \( \mathbb{Q} \) on \( \mathbb{R} \), is not continuous at any \( t \in \mathbb{R} \). □

**Example 2.2.** Let \( S \) be a countably infinite subset of \( \mathbb{R} \) which we enumerate as \( \{ t_1, t_2, \ldots \} \). Define the self-map \( f \) on \( \mathbb{R} \) by

\[
 f(t) := \sum_{i \in I(t)} 2^{-i} \quad \text{where} \quad I(t) := \{ i \in \mathbb{N} : a_i \leq t \}.
\]

It is not difficult to check that \( f \) is a right-continuous and increasing map which is discontinuous precisely at the points that belong to \( S \). (Exercise!) Conclusion: For any countable subset \( S \) of \( \mathbb{R} \), there is a right-continuous and increasing map which is continuous everywhere but at the points of \( S \). □

**Remark 2.2.** (Weierstrass’ Theorem on \( \mathbb{R} \)) A continuous real map \( f \) on an interval of the form \( [a, b] \) attains its maximum and minimum. The second part of this claim follows from applying the first one to \(- f\), so it is enough to prove only the first claim. Put \( s := \sup f ([a, b]) \). Then, by Exercise 1.5, there is a sequence \( (t_m) \) in \( [a, b] \) such that \( f(t_m) \uparrow s \). By the Bolzano-Weierstrass Theorem, there is a subsequence \( (t_{m_k}) \) of \( (t_m) \) such that \( t_{m_k} \to t \) for some \( t \in [a, b] \). By continuity of \( f \) at \( t \), then, \( f(t_{m_k}) \to f(t) \), and hence \( f(t) = s \), and our claim is proved. □

**Remark 2.3.** (The Intermediate Value Theorem on \( \mathbb{R} \)) For any continuous real map \( f \) on an interval of the form \([a, b]\) such that \( f(a) < f(b) \), we have \( f([a, b]) \subseteq f((a, b)) \). To see this, take any real number \( c \) with \( f(a) < c < f(b) \), and put \( s := \sup \{ t \in [a, b] : f(t) < c \} \). Now use Exercise 1.5 and continuity of \( f \) to get \( f(s) \leq c \), while by definition of \( s \), we cannot have \( f(s+) < c \). Thus, \( f(s) = c \), and our claim is proved. □

**Remark 2.4.** Any continuous real function on \( \mathbb{R} \) is identified by its values on \( \mathbb{Q} \). That is, if \( f \) and \( g \) are two continuous self-maps on \( \mathbb{R} \) such that \( f|_\mathbb{Q} = g|_\mathbb{Q} \), then \( f = g \). Indeed, for any \( t \in \mathbb{R} \), there is a sequence \( (r_m) \) of rational numbers with \( r_m \to t \) (Proposition 1.2), and hence, by continuity of \( f \) and \( g \), and because these functions agree on \( \mathbb{Q} \), we find \( f(t) = \lim f(r_m) = \lim g(r_m) = g(t) \). □

**Application:** \( \mathbb{R} \approx \text{card} \ C(\mathbb{R}) \).

We have proved in Chapter A that \( \mathbb{R} \approx \text{card} \ \{0, 1\}^\mathbb{N} \) and \( \mathbb{Q} \approx \text{card} \ \mathbb{N} \). By Exercise A.3.8, therefore, \( \mathbb{R}^\mathbb{Q} \approx \text{card} \ \mathbb{R}^\infty \approx \text{card} \ \{0, 1\}^{\text{card} \ \mathbb{N}} \). But, in fact, we also have \( \{0, 1\}^\mathbb{N} \approx \text{card} \ \{0, 1\}^\infty \). (Proof. Represent any one element of \( \{0, 1\}^\infty \) as a sequence of binary sequences, and define \( f : (\{0, 1\}^\infty)^\mathbb{N} \to \{0, 1\}^\infty \) as

\[
 f((a^1_m), (a^2_m), \ldots) := (a^1_1, a^2_1, a^1_2, a^2_1, a^1_3, a^2_2, \ldots).
\]
It is readily checked that \( f \) is a bijection.) It follows that \( \mathbb{R}^Q \approx_{\text{card}} \mathbb{R}^N \approx_{\text{card}} \{0,1\}^\infty \approx_{\text{card}} \mathbb{R} \).

But, in view of Remark 2.4, we may define a map \( G : C(\mathbb{R}) \to \mathbb{R}^Q \) by \( G(\varphi) := \varphi|_Q \). It is readily checked that \( G \) is a injection, and it follows that \( \mathbb{R}^Q \supseteq_{\text{card}} C(\mathbb{R}) \). On the other hand, \( C(\mathbb{R}) \supseteq_{\text{card}} \mathbb{R} \) – for instance, mapping any real number \( x \) to the constant function on \( \mathbb{R} \) that equals \( x \) everywhere yields an injection from \( \mathbb{R} \) into \( C(\mathbb{R}) \). Conclusion: \( \mathbb{R} \approx_{\text{card}} \mathbb{R}^Q \approx_{\text{card}} C(\mathbb{R}) \approx_{\text{card}} \mathbb{R} \), and hence \( \mathbb{R} \supseteq_{\text{card}} C(\mathbb{R}) \supseteq_{\text{card}} \mathbb{R} \). Invoking the Schröder-Bernstein Theorem (Section A.3.1) completes our proof. \( \square \)

Uniform Continuity of a Real Map on an Interval

A real map \( f \) on \( I \) is said to be **uniformly continuous**, if for each \( \varepsilon > 0 \), there is a \( \delta > 0 \) such that \( |f(s) - f(t)| < \varepsilon \) for every \( s, t \in T \) with \( |s - t| < \delta \). While continuity is a “local” phenomenon, uniform continuity is a “global” property that says that whenever any two points in the domain of the function are close to each other, so should the values of the function at these points.

It is plain that a uniformly continuous real map is continuous. The converse is false, however. For instance, the map \( t \mapsto 1/t \) on \((0,1)\) is continuous, but not uniformly continuous. There is, however, one important case in which uniform continuity and continuity coincide.

**Proposition 2.1.** (Heine) For any real numbers \( a \) and \( b \) with \( a \leq b \), a map \( f \in \mathbb{R}^{[a,b]} \) is continuous if, and only if, it is uniformly continuous.

**Proof.** To derive a contradiction, assume that \( f \) is continuous, but not uniformly so. Then, for some \( \varepsilon > 0 \), we can find two sequences \((t_m, s_m)\) in \([a,b]\) with \( |t_m - s_m| < 1/m \) and \( |f(t_m) - f(s_m)| \geq \varepsilon \), for each positive integer \( m \). By the Bolzano-Weierstrass Theorem, there exists a convergent subsequence \((t_{m_k})\) of \((t_m)\). Put \( t := \lim t_{m_k} \) and note that \( t \) belongs to \([a,b]\). Moreover, since \( |t_{m_k} - s_{m_k}| \to 0 \) as \( k \to \infty \), we must have \( s_{m_k} = t \). Therefore, \( \lim f(t_{m_k}) = f(t_{m_k}) \), which entails \( |f(t_M) - f(s_M)| < \varepsilon \) for some \( M \in \mathbb{N} \) large enough, contradicting our choice of the sequences \((t_m)\) and \((s_m)\). \( \square \)

Preservation of Continuity under Addition, Multiplication and Composition

Let \( I \) be a nondegenerate interval and \( t \in I \). If the limits of \( f, g \in \mathbb{R}^I \) at \( t \) exist and finite, then we have

\[
\lim_{s \to t} (f(s) + g(s)) = \lim_{s \to t} f(s) + \lim_{s \to t} g(s)
\]

and

\[
\lim_{s \to t} f(s)g(s) = \lim_{s \to t} f(s) \lim_{s \to t} g(s).
\]

If \( \lim_{s \to t} f(s) = \infty \), then these formulas remain valid provided that \( \lim_{s \to t} g(s) \neq -\infty \) and \( \lim_{s \to t} g(s) \neq 0 \), respectively. \( \square \)

For any \( m \in \mathbb{N} \), a **polynomial function of degree** \( m \) on \( I \) is a map \( f \in \mathbb{R}^I \) such that

\[
f(t) = a_0 + a_1 t + \cdots + a_m t^m \quad \text{for every } t \in I
\]

for some real numbers \( a_0, \ldots, a_m \) with \( a_m \neq 0 \). The set of all constant real maps and polynomial functions (of any degree) on \( I \) is denoted as \( \mathbf{P}(I) \), but, again, if \( I \) is an interval of the form \([a,b]\), we write \( \mathbf{P}[a,b] \) instead of \( \mathbf{P}([a,b]) \). \( \square \)

Finally, suppose \( h \) is a continuous real map on an interval that contains \( f(I) \). Then, it is easily verified that \( h \circ f \) is continuous, provided that \( f \) is continuous. This fact and part (a) of Exercise 1.2 ensure that \( |f| \in \mathbf{C}(I) \) for every \( f \in \mathbf{C}(I) \).
Continuity of Monotonic Real Functions

Let \( I \) be a nondegenerate interval and \( f \) a real map on \( I \). We say that \( t \in I \) is a point of discontinuity of \( f \) if \( f \) is not continuous at \( t \). To be sure, a real function may well have uncountably many points of discontinuity. (Example. Consider the indicator function of \( \mathbb{Q} \) on \( \mathbb{R} \).) However, the situation is much more pleasant for monotonic functions.

**Proposition 2.2.** Let \( I \) be a nondegenerate interval, and \( f \in \mathbb{R}^I \). If \( f \) is monotonic, then it can have at most countably many points of discontinuity.

**Proof.** Let \( f \) be increasing, and take any \( t \in I \) that is not an endpoint of \( I \). Define

\[
g(t) := \sup \{ f(s) : s \in I \text{ and } s < t \}
\]

and

\[
h(t) := \inf \{ f(s) : s \in I \text{ and } s > t \}
\]

for any \( t \in I \). We leave it to you to show that \( f(t-) = g(t) \) and \( f(t+) = h(t) \) for every \( t \in I \). On the other hand, if \( t \) is a point of discontinuity of \( f \), then \( f(t-) < f(t+) \). (Yes?) It follows that, for any such \( t \), \( J_t := [g(t), h(t)] \) is a nondegenerate interval. Moreover, if \( t \) and \( t' \) are any two such elements of \( I \) with \( t < t' \), then it is easily checked that \( h(t) \leq g(t') \), thanks to the fact that \( f \) is increasing. It follows that the collection of all \( J_t \), where \( t \) varies over the set of all points of discontinuity of \( f \) that do not equal an endpoint of \( I \), is a collection of nondegenerate intervals any two of which overlap at most at one point. As there can be at most countably such intervals (Corollary A.3.6), we are done. \( \blacksquare \)

### 2.2 Differentiation of Real Maps on \( \mathbb{R} \)

**Derivatives of a Real Map on an Interval**

Let \( I \) be a nondegenerate interval, and take any real function \( f \) on \( I \). For any given \( t \in I \), we define the difference-quotient map \( Q_{f,t} : I \setminus \{t\} \to \mathbb{R} \) by

\[
Q_{f,t}(s) := \frac{f(s) - f(t)}{s - t}.
\]

If \( t \) is not a right endpoint of \( I \), and the right-limit of this map at \( t \) equals a real number, then \( f \) is said to be right-differentiable at \( t \). In this case, the number \( Q_{f,t}(t+) \) is called the right-derivative of \( f \) at \( t \), and is denoted by \( f'_+(t) \). Similarly, if \( t \) is not a left endpoint of \( I \) and \( Q_{f,t}(t-) \in \mathbb{R} \), then \( f \) is said to be left-differentiable at \( t \), and the left-derivative of \( f \) at \( t \), which we denote by \( f'_-(t) \), is defined as the number \( Q_{f,t}(t-) \). If \( t \) is the left endpoint of \( I \) and \( f'_-(t) \) exists, or if \( t \) is the right endpoint of \( I \) and \( f'_+(t) \) exists, or if \( t \) is not an endpoint of \( I \) and both \( f'_+(t) \) and \( f'_-(t) \) exist and we have \( f'_+(t) = f'_-(t) \), we say that \( f \) is differentiable at \( t \). In the first case \( f'_+(t) \), in the second case \( f'_-(t) \), and in the third case the common value of \( f'_+(t) \) and \( f'_-(t) \), is denoted as \( f'(t) \). When it exists, the number \( f'(t) \) is called the derivative of \( f \) at \( t \). Evidently, if \( t \) is not an endpoint of \( I \), then \( f \) is differentiable at \( t \) iff

\[
\lim_{s \to t} \frac{f(s) - f(t)}{s - t} \in \mathbb{R},
\]

in which case \( f'(t) \) equals precisely to this number. If \( J \) is an interval contained in \( I \), and \( f \) is differentiable at each \( t \in J \), we say that \( f \) is differentiable on \( J \). If \( J = I \) here, then we simply say that \( f \) is differentiable. In that case the derivative of \( f \) is defined as the function \( f' : I \to \mathbb{R} \) that maps each \( t \in I \) to the derivative of \( f \) at \( t \). (If \( f' \) is differentiable, then \( f \) is said to be twice differentiable, and the second derivative of \( f \) is defined as the function \( f'' \in \mathbb{R}^I \) that maps each \( t \in I \) to the derivative of \( f' \) at \( t \).) Similarly, if \( f \) is right-differentiable at each \( t \in I \), then it is said…
to be right-differentiable, and in this case we define the right-derivative of $f$ as a real function on $I$ that maps every $t \in I$ to $f'_+(t)$. Naturally, this function is denoted as $f'_+$. Left-differentiability of $f$ and the function $f'_-$ are analogously defined.

**Exercise 2.1.** Let $I$ be an open interval and take any $f$ and $g$ in $\mathbb{R}^I$.

(a) Show that if $f$ is both right- and left-differentiable at $t \in I$, then it is continuous at $t$.

(b) Show that if $f$ and $g$ are differentiable, and $a \in \mathbb{R}$, then $af + g$ and $fg$ are differentiable. Conclude that every $f \in \mathcal{P}(I)$ is differentiable.

(c) (The Chain Rule) Let $f$ be differentiable and $h$ is a real map on an interval that contains $f(I)$. Show that if $h$ is differentiable on $f(I)$, then so is $h \circ f$ and $(h \circ f)' = (h' \circ f)f'$.

Take any real numbers $a$ and $b$ with $a < b$. For any $f \in C[a, b]$, our definition maintains that the derivatives of $f$ at $a$ and $b$ are $f'_+(a)$ and $f'_+(b)$, respectively, provided that they exist. Thus $f$ being differentiable means that $f[a, b]$ is right-differentiable, $f|_{[a, b]}$ is left-differentiable, and $f'_+(t) = f'_-(t)$ for each $t$ in $(a, b)$. If $f' \in C[a, b]$, then we say that $f$ is **continuously differentiable** – the collection of all such maps is denoted by $C^1[a, b]$. If, further, $f' \in C^1[a, b]$, then $f$ is **twice continuously differentiable** – the collection of all such maps is denoted by $C^2[a, b]$. For any positive integer $k$, the collection $C^k[a, b]$ is defined inductively. Finally, we let $C^k[a, \infty)$ stand for the collection of all $f \in C[a, \infty)$ with $f|_{[a, b]} \in C^k[a, b]$ for every $b > a$. (The collections $C^k(-\infty, b]$ and $C^k(\mathbb{R})$ are defined analogously.)

**The Mean Value Theorem**

Let $a$ and $b$ be any two real numbers with $a < b$. Denote the open interval $(a, b)$ by $I$, and take any differentiable real map $f$ on $I$. If $f$ attains its maximum, that is, $f(t) = \max f(I)$ for some $t \in I$, then $f'(t) = 0$. (Proof. If $f'(t) > 0$ (or $< 0$), then we can find a small enough $\varepsilon > 0$ ($< 0$, respectively) such that $a < t + \varepsilon < b$ and $f(t) < f(t + \varepsilon)$, contradicting that $f(I) = f(t)$.) Of course, the same would be true, if $f$ assumed instead its minimum at $t$. (Proof. Apply the previous observation to $-f$.) These simple facts are the reasons behind quite a few of the classical theorems of differential calculus.

**Rolle’s Theorem.** Let $a$ and $b$ be two real numbers with $a < b$, and take any $f \in C[a, b]$ that is differentiable on $(a, b)$. If $f(a) = f(b)$, then $f'(c) = 0$ for some real number $c$ in $(a, b)$.

**Proof.** Since $f$ is continuous, there exist $t_*, t^* \in [a, b]$ such that $f(t_*) \leq f(t) \leq f(t^*)$ for every $t$ in $[a, b]$ (Remark 2.2). Now assume that $f(a) = f(b)$. If $\{t_*, t^*\} \subseteq \{a, b\}$, then $f$ must be a constant function, and hence $f'(t) = 0$ for all $a \leq t \leq b$. If this is not the case, then either $t_* \in (a, b)$ or $t^* \in (a, b)$. In the former case we have $f'(t_*) = 0$ (because $f$ attains its minimum at $t_*$), and in the latter case $f'(t^*) = 0$.

**The Mean Value Theorem.** (Cauchy) Let $a$ and $b$ be two real numbers with $a < b$, and take any $f \in C[a, b]$ that is differentiable on $(a, b)$. Then

$$f(b) - f(a) = f'(c)(b - a)$$

for some real number $c$ in $(a, b)$.

**Proof.** Apply Rolle’s Theorem to the map $t \mapsto f(b) - f(t) - \left(\frac{b - a}{c - a}\right)(f(b) - f(a))$ on $[a, b]$.

**Exercise 2.2.** Let $a$, $b$, and $f$ be as in the Mean Value Theorem. Show that $f' \geq 0$ iff $f$ is increasing, while $f' > 0$ implies that it is strictly increasing, but not conversely.
Lipschitz Continuity of Continuously Differentiable Real Maps

Recall that a function \( f : [a, b] \to \mathbb{R} \) is said to be Lipschitz continuous if there is real number \( K > 0 \) such that
\[
|f(s) - f(t)| \leq K |s - t| \quad \text{for every } s, t \in [a, b].
\]  

(4)

A useful consequence of the Mean Value Theorem is that any differentiable map on an interval with bounded derivatives is Lipschitz continuous. This fact is usually used in the following form.

**Corollary 2.3.** Every \( f \in C^1[a, b] \) is Lipschitz continuous.

**Proof.** By Remark 2.2, \( K := \sup \{|f'(t)| : t \in [a, b]\} \) is a real number. Moreover, by the Mean Value Theorem, \( |f(s) - f(t)| \leq K |s - t| \) for every \( s, t \in (a, b) \). It is easy to show that (4) follows from this fact and continuity of \( f \).

Taylor’s Theorem

The idea behind the Mean Value Theorem can be used to get a better approximation theorem for smoother maps. The following is extremely useful in this regard.

**Taylor’s Theorem.** Let \( a \) and \( b \) be two real numbers with \( a < b \), and take any positive integer \( k \) and a map \( f \in C^k[a, b] \) such that \( f^{(k)} \) is differentiable on \( (a, b) \). Then, for any \( t_o \) and \( t \) in \([a, b]\),
\[
f(t) = f(t_o) + f'(t_o)(t - t_o) + \frac{f''(t_o)}{2!}(t - t_o)^2 + \cdots + \frac{f^{(k+1)}(c)}{(k+1)!}(t - t_o)^{k+1}
\]
for some \( c \) between \( t_o \) and \( t \).

**Proof.** Let \( I \) be the closed interval with endpoints \( t_o \) and \( t \). Define \( g \in C^k(I) \) by
\[
g(s) := f(t) - f(s) - \sum_{i \in [k]} \frac{f^{(i)}(s)}{i!}(t - s)^i - \frac{M}{(k + 1)!}(t - s)^{k+1},
\]
where the real number \( M \) is chosen to guarantee that \( g(t_o) = 0 \). Clearly, \( g \) is differentiable on \( I \setminus \{t_o, t\} \), and a straightforward computation (using Exercise 2.1) yields \( g'(s) = \frac{1}{k!}(t - s)^k(M - f^{(k+1)}(s)) \) for every \( s \in I \setminus \{t_o, t\} \). Moreover, since \( g(t_o) = 0 = g(t) \) and \( g \in C(I) \), Rolle’s Theorem guarantees that \( g'(c) = 0 \) for some \( c \in I \setminus \{t_o, t\} \). But then \( M = f^{(k+1)}(c) \), and substituting this finding in the equation \( g(t_o) = 0 \) proves the claim.

2.3 Concave and Convex Functions on \( \mathbb{R} \)

**Definitions**

Let \( n \) be a positive integer, and recall that a subset \( S \) of \( \mathbb{R}^n \) is said to be convex if it contains the line segment between any two of its members, that is, \( \lambda s + (1 - \lambda)t \in S \) for every \( s, t \in S \) and \( 0 \leq \lambda \leq 1 \). Given any such nonempty set \( S \), a map \( f \in \mathbb{R}^S \) is said to be concave if
\[
f(\lambda s + (1 - \lambda)t) \geq \lambda f(s) + (1 - \lambda)f(t)
\]
for every \( s, t \in S \) and \( 0 \leq \lambda \leq 1 \), and strictly concave if this inequality holds strictly for every distinct \( s, t \in S \) and \( 0 < \lambda < 1 \). The definition of convex and strictly convex functions are obtained by reversing these inequalities. Equivalently, \( f \) is called (strictly) convex if \(-f\) is (strictly) concave. Finally, \( f \) is said to be affine if it is both concave and convex.
Clearly, there is a proof. Our assertion follows.

Take any proof. But countably many points of $I$ are concave maps, then it is easily checked that so is $g \circ f$.

**FIGURE 1 ABOUT HERE**

We now specialize to concave functions defined on an interval $I \subseteq \mathbb{R}$. The following result, which is illustrated in Figure 1, teaches a lot about such functions.

**Lemma 2.4.** Let $I$ be an interval and $f \in \mathbb{R}^I$ a concave map. Then,

$$
\frac{f(s_2) - f(t)}{s_2 - t} \leq \frac{f(s_1) - f(t)}{s_1 - t} \leq \frac{f(r_1) - f(t)}{r_1 - t} \leq \frac{f(r_2) - f(t)}{r_2 - t}
$$

for any $r_2 \leq r_1 < t < s_1 \leq s_2$ with $r_2, s_2 \in I$.

**Proof.** Clearly, there is a $\lambda$ in $(0, 1]$ such that $r_1 = \lambda r_2 + (1 - \lambda)t$, so, by concavity, $\lambda f(r_2) + (1 - \lambda)f(t) \leq f(r_1)$, that is, $f(r_2) - f(t) \leq \frac{1}{\lambda}(f(r_1) - f(t))$. Combining this with the fact that $\lambda = \frac{r_1 - t}{r_2 - t}$ yields the right-most inequality. The other two inequalities are proved similarly.

Differentiability of Concave Functions on $\mathbb{R}$

Here is the basic result about the differentiability of concave functions:

**Proposition 2.5.** (Stolz) Let $I$ be an open interval and $f \in \mathbb{R}^I$ a concave map. Then, $f$ is both right- and left-differentiable, and both $f_+’$ and $f_-’$ are decreasing functions on $I$. Moreover, $f_+’ \leq f_-’$ and $f_-’(t_1) \leq f_+’(t_2)$ for any $t_1, t_2 \in I$ with $t_2 < t_1$.

**Proof.** The right- and left-differentiability of $f$ follow from Lemma 2.4 and Proposition 1.4. On the other hand, for any $t$ in $I$, we may apply Lemma 2.4 to find

$$
f_+’(t) = \lim_{s_1 \downarrow t} \frac{f(s_1) - f(t)}{s_1 - t} \leq \frac{f(r_1) - f(t)}{r_1 - t}
$$

for any $r_1 < t$ with $r_1 \in I$. Applying Lemma 2.4 one more time,

$$
f_+’(t) \leq \lim_{r_1 \downarrow t} \frac{f(r_1) - f(t)}{r_1 - t} = f_+’(t),
$$

so, in view of the arbitrary choice of $t$, we may conclude that $f_+’ \leq f_-’$. The other assertions of the proposition are proved by using Lemma 2.4 analogously.

**Corollary 2.6.** Let $I$ be an open interval and $f \in \mathbb{R}^I$ a concave map. Then, $f$ is differentiable at all but countably many points of $I$.

**Proof.** Take any $t \in I$ and any sequence $(t_m)$ in $I \cap (-\infty, t)$ with $t_m \uparrow t$. By Proposition 2.5, we have $f_+’(t_m) \geq f_-’(t) \geq f_+’(t)$ for each $m$. Letting $m \to \infty$, therefore, we find $f_+’(t) = f_-’(t)$, provided that $f_+’$ is continuous at some $t$. Conclusion: $f$ is differentiable on every $t \in I$ at which $f_+’$ is continuous. (The converse of this is true as well.) But, again by Proposition 2.5, $f_+’$ is a decreasing map on $I$. As such a function is continuous at all but countably many points in $I$ (Proposition 2.2), our assertion follows.

Continuity of Concave Functions on $\mathbb{R}$
We have seen in Exercise 2.1.(a) that the existence of right- and left-derivatives of a real function at a given point in an open interval implies that the continuity of that function at that point. The following observation is thus an immediate consequence of Proposition 2.5.

**Corollary 2.7.** (Jensen) Let $I$ be an open interval and $f \in \mathbb{R}^I$. If $f$ is concave (or convex), then it is continuous.

In view of Remark 2.2, then:

**Corollary 2.8.** For any real number $a$ and $b$ with $a \leq b$, if $f \in \mathbb{R}^{[a,b]}$ is concave (or convex), then $\inf f([a,b]) > -\infty$ and $\sup f([a,b]) < \infty$.

**A Supporting Hyperplane Theorem**

Another useful consequence of Proposition 2.5 is illustrated in Figure 2. This is:

**Corollary 2.9.** Let $I$ be an open interval and $f \in \mathbb{R}^I$ a concave map. Then, for every $t$ in $I$, there is an $(a,b) \in \mathbb{R}^2$ such that

$$at + b = f(t) \quad \text{and} \quad f(s) \leq as + b \quad \text{for all } s \in I.$$

**Proof.** Thanks to Proposition 2.5, we can find a real number $a$ such that $f'_+(t) \leq a \leq f'_-(t)$. In turn, we set $b := f(t) - at$. Now, take any $s \in I$. If $t < s$, then, by Lemma 2.4,

$$\frac{f(s) - f(t)}{s - t} \leq f'_+(t) \leq a,$$

and hence $f(s) \leq as + b$. The argument for the case $s < t$ is similar. 

Finally, we recall that, provided that $f$ is differentiable on an open interval $I$, then it is (strictly) concave iff $f'$ is (strictly) decreasing. Provided that $f$ is twice differentiable, it is concave iff $f'' \leq 0$, while $f'' > 0$ implies the strict concavity of $f$. (The converse of the latter statement is false. Why?) These facts are easily proved by means of Taylor’s Theorem.

**Exercise 2.3.** Let $I$ be an interval, and $f \in \mathbb{R}^I$ a strictly increasing bijection. Prove: $f$ is (strictly) concave iff $f^{-1}$ is (strictly) convex.

**Exercise 2.4.** For any given positive integer $n$, let $S$ be a nonempty convex subset of $\mathbb{R}^n$, and $\mathcal{F}$ a collection of concave functions in $\mathbb{R}^S$. Show that if $\inf \{f(\omega) : f \in \mathcal{F}\} > -\infty$ for every $\omega \in S$, then the map $\omega \mapsto \inf \{f(\omega) : f \in \mathcal{F}\}$ is a concave real map on $S$.

**Exercise 2.5.** For any given positive integer $n$, let $S$ be a nonempty convex subset of $\mathbb{R}^n$, and $(f_m)$ a sequence of concave functions in $\mathbb{R}^S$. Show that if $\lim f_m(t)$ is a real number for each $t \in S$, then the map $t \mapsto \lim f_m(t)$ is a concave real map on $S$.

**Exercise 2.6.** Show that a concave real map on an open interval $I$ is differentiable at $t \in I$ iff $f'_+ \text{ is continuous at } t$.

**Exercise 2.7.** Let $I$ be an open interval and $f$ a concave real map on $I$. Prove: For any $a, b \in I$, there is a real number $K > 0$ such that $|f(s) - f(t)| \leq K |s - t|$ for each $s, t \in [a,b]$. 

17
Exercise 2.8. Define the self-map $f$ on $\mathbb{R}$ by $f(t) := -|t| - \sum_{i=1}^{\infty} 2^{-i} |t - i|$. Show that $f$ is not differentiable at infinitely many points.

Exercise 2.9. (Saks) Show that a self-map $f$ on $\mathbb{R}$ is concave if

$$\inf(f + g)([a, b]) = \min\{f(a) + g(a), f(b) + g(b)\}.$$

for every real numbers $a$ and $b$ with $a < b$ and every affine self-map $g$ on $\mathbb{R}$.

### 2.4 Quasiconcave and Quasiconvex Functions

With $S$ being a nonempty convex subset of $\mathbb{R}^n$, $n \in \mathbb{N}$, we say that a function $f \in \mathbb{R}^S$ is **quasiconcave** if

$$f(\lambda s + (1 - \lambda)t) \geq \min\{f(s), f(t)\} \quad \text{for any } s, t \in S \text{ and } 0 \leq \lambda \leq 1,$$

and **strictly quasiconcave** if this inequality holds strictly for any distinct $s, t \in S$ and $0 < \lambda < 1$. ($f$ is called **strictly quasiconvex** if $-f$ is (strictly) quasiconcave.) It is easy to show that $f$ is quasiconcave iff $f^{-1}([a, \infty))$ is a convex set for any real number $a$. Besides, every concave function in $\mathbb{R}^T$ is quasiconcave, but not conversely. For instance, if $S \subseteq \mathbb{R}$, then every monotonic function in $\mathbb{R}^S$ is quasiconcave, but of course, every monotonic function in $\mathbb{R}^S$ is not concave.

Quasiconcavity plays an important role in optimization theory, and it is often invoked to establish the uniqueness of a solution for a maximization problem. Indeed, if $f \in \mathbb{R}^S$ is strictly quasiconcave, and there exists an $s \in S$ with $f(s) = \max f(S)$, then $s \in S$ must be the only element of $S$ with this property. For, if $s \neq t$ and $f(t) = \max f(S)$, then $f(s) = f(t)$, so $f(\frac{s + t}{2}) > f(s) = \max f(S)$ by strict quasiconcavity. Since $s, t \in S$ and $S$ is convex, this is impossible.